8

Radiating Flows

In a radiating fluid, radiation affects the energy and momentum balance in the flow, and can drive the thermodynamic state of the material out of equilibrium. In this chapter we consider a few interesting examples of radiating-flow problems in both the linear and nonlinear regimes.

For small-amplitude disturbances we focus on radiative energy and momentum exchange; we first examine the radiative smoothing of temperature fluctuations in a static medium, then the effects of radiation on acoustic waves in a homogeneous medium, and finally the effects of radiation on acoustic-gravity waves in a stratified medium. We shall draw our examples primarily from astrophysics where considerable attention has been given to radiative effects on the propagation and dissipation of waves in the atmospheres of the Sun and stars.

For nonlinear disturbances we meet a much richer variety of phenomena. We consider first the conceptually simple problem of penetration of radiation into a passive static medium as a thermal wave. We then examine the effects of radiative transport across steady shocks and in propagating shocks in both the weak- and strong-shock limits, including the case of a propagating non-LTE shock, where radiation determines the state of the material. We next examine the interplay of radiation and hydrodynamics in propagating ionization fronts. Finally, we consider the dynamics of radiation-driven stellar winds, where the primary effect of radiation is on the momentum balance in the flow.

The reader should note that in most of the applications to be discussed the treatment of radiation falls far below the standards set in Chapter 7, although reasonably complete and consistent solutions are obtained for one or two simple problems. We make this remark not as a criticism of the existing literature, but rather to call attention to the rewarding opportunities that exist for new research exploiting the more complete formulation of the dynamical behavior of radiation that is now available.

8.1 Small-Amplitude Disturbances

100. Radiative Damping of Temperature Fluctuations

Valuable insight into the effects of radiative energy exchange on small-amplitude disturbances can be obtained by examining the smoothing of temperature fluctuations in a radiating fluid.

QUASI-STATIC RADIATION TRANSPORT

Consider a field of temperature perturbations imposed on a static, grey, LTE ambient medium, initially in radiative equilibrium. We assume there are no fluid motions ($\mathbf{v}\equiv 0$), and that heat is exchanged only radiatively. Under these assumptions the gas energy equation (96.7) reduces to

$$\rho(\partial e/\partial t) = 4\pi\kappa(J - B), \tag{100.1}$$

where, from (52.21), $(\partial e/\partial t) = c_v(\partial T/\partial t)$ in a static medium. The thermal source is $B = (a_R c/4\pi)T^4$, and the mean intensity is $4\pi J = \oint I\, d\omega$. We assume that the characteristic time scale associated with the disturbances is so long that the radiation field can be taken to be quasi-static. Then I is given by the static transfer equation

$$(\partial I/\partial s) = \kappa(B - I), \tag{100.2}$$

where s is the path length along a ray. In using (100.2) we neglect all dynamical effects of the radiation field.

For small disturbances we linearize, writing

$$T(\mathbf{x}, t) = T_0(\mathbf{x}) + T_1(\mathbf{x}, t), \tag{100.3}$$

$$B(\mathbf{x}, t) = B_0(\mathbf{x}) + (a_R c T_0^3/\pi)T_1 \equiv B_0 + B_1, \tag{100.4}$$

and

$$\kappa(\mathbf{x}, t) = \kappa_0(\mathbf{x}) + (\partial\kappa_0/\partial T)T_1 \equiv \kappa_0 + \kappa_1. \tag{100.5}$$

The linearized energy equation is then

$$\rho c_v(\partial T_1/\partial t) = 4\pi\kappa_0(J_1 - B_1) + 4\pi\kappa_1(J_0 - B_0), \tag{100.6}$$

where J_1 is the local perturbation of the mean intensity induced by perturbations in the source–sink terms throughout the medium. Because we assume that the material is initially in radiative equilibrium, $J_0 \equiv B_0$, hence the term containing κ_1 in (100.6) vanishes identically.

J_1 is the angle average of I_1, the local change in the specific intensity, which can be calculated from the linearized transfer equation

$$(\partial I_1/\partial s) = \kappa_0(B_1 - I_1) + \kappa_1(B_0 - I_0). \tag{100.7}$$

If we now make the simplifying assumptions that the ambient medium is *homogeneous* and of *infinite extent* (appropriate for a study of, say, pure acoustic waves), then the unperturbed radiation field will be *isotropic*, which implies that $I_0 \equiv J_0 \equiv B_0 =$ constant. Hence the term containing κ_1 in (100.7) vanishes identically. Thus in a homogeneous medium, both the energy equation and the transfer equation are unaffected (to first order) by a perturbation in the opacity.

I_1 is found directly from the formal solution of (100.7):

$$I_1(\mathbf{x}_0, \mathbf{n}) = \int_0^\infty B_1(\mathbf{x}_0 - \mathbf{n}s)e^{-\kappa_0 s}\kappa_0\, ds. \tag{100.8}$$

The energy equation for a field of temperature perturbations in an infinite, homogeneous, static medium is thus

$$[\partial T_1(\mathbf{x}_0)/\partial t] = -\nu\left[T_1(\mathbf{x}_0) - (4\pi)^{-1}\oint d\omega \int_0^\infty T_1(\mathbf{x}_0 - \mathbf{n}s)e^{-\kappa_0 s}\kappa_0\, ds\right], \tag{100.9}$$

where

$$\nu \equiv 4a_R c\kappa_0 T_0^3/\rho c_v = 16\sigma_R \kappa_0 T_0^3/\rho c_v \tag{100.10}$$

is an inverse time scale characterizing the rate of energy loss by radiative emission in the absence of reabsorption.

Suppose now we have a field of planar temperature disturbances varying as $e^{i\mathbf{k}\cdot\mathbf{x}}$. Choose $\mathbf{k}$ as a preferred direction defining the x axis (which is otherwise arbitrary in a homogeneous medium), and let $\cos^{-1}\mu$ be the angle between $\mathbf{k}$ and $\mathbf{n}$. We can then rewrite (100.9) as

$$[\partial T_1(x_0)/\partial t] = -\nu\left[T_1(x_0) - \tfrac{1}{2}\int_{-1}^{1} d\mu \int_0^\infty T_1(x_0 \mp \xi)e^{-\kappa_0 \xi/|\mu|}\kappa_0\, d\xi/|\mu|\right], \tag{100.11}$$

where the sign in the argument of the integrand is chosen opposite to the sign of μ.

Equation (100.11) admits separable solutions of the form

$$T_1(x, t) = \phi(k, t)e^{ik(x-x_0)}. \tag{100.12}$$

Note that because (100.11) is linear, any linear combination of solutions of the general form (100.12) will satisfy (100.11), hence we can synthesize the behavior of an arbitrary field of fluctuations by a suitable superposition of its Fourier components. Using (100.12) in (100.11) we have

$$(\partial\phi/\partial t) = -n(k)\phi, \tag{100.13}$$

where, by virtue of symmetry considerations that simplify the integral,

$$n(k) = \nu\left[1 - \int_0^1 d\mu \int_0^\infty \cos(\mu k y/\kappa_0)e^{-y}\, dy\right]. \tag{100.14}$$

Thus a spatially harmonic temperature disturbance with wavenumber k decays exponentially from its initial value $\phi(k, 0)$ according to

$$\phi(k, t) = \phi(k, 0)\exp[-t/t_{RR}(k)] \tag{100.15}$$

where the *radiative relaxation time* is $t_{RR}(k) = 1/n(k)$. It is clear on physical grounds that n must always be positive because in the situation we are considering $|J_1|$ is always less than or equal to $|B_1|$ (equality occurring only in the limit of infinite optical thickness), and therefore regions of enhanced temperature always tend to cool while cooler regions tend to heat, thus damping the disturbance.

Table 100.1. Radiative Relaxation Rates and Eddington Factors

κ_0/k	$n(k)/\nu$	$n_E(k)/\nu$	$f(k)$
0.0	1.000	1.000	0.000
0.2	0.725	0.893	0.106
0.4	0.524	0.676	0.176
0.6	0.382	0.481	0.222
0.8	0.283	0.342	0.253
1.0	0.215	0.250	0.273
1.5	0.118	0.129	0.301
2.0	0.073	0.076	0.314
5.0	0.013	0.013	0.330
∞	0.000	0.000	0.333

From standard tables one has

$$\int_0^\infty \cos(\mu k y/\kappa_0)e^{-y}\,dy = [1+(k/\kappa_0)^2\mu^2]^{-1}, \qquad (100.16)$$

which gives the angular distribution of $I_1(\mu)$ for a given (k/κ_0). Using (100.16) in (100.14) we obtain the dispersion relation for the *thermal relaxation mode* of a radiating fluid:

$$n(k) = \nu[1-(\kappa_0/k)\cot^{-1}(\kappa_0/k)], \qquad (100.17)$$

a result first obtained by Spiegel (**S18**).

The ratio $\kappa_0/k = \kappa_0\Lambda/2\pi = \tau_\Lambda/2\pi$, where τ_Λ is the optical thickness of one wavelength of the perturbation. As shown in Table 100.1, $n(k)$ decreases monotonically from ν to zero as τ_Λ varies from zero to infinity; that is, optically thin disturbances damp rapidly whereas optically thick disturbances damp slowly. To understand this result intuitively one notes that in an optically thin perturbation $I_1 \to 0$ because positive and negative fluctuations of T_1 contribute equally along each line of sight, and average to zero. In this case $J_1 \to 0$, hence the damping rate is set entirely by local emission losses, independently of k. In contrast, in an optically thick perturbation $J_1 \approx B_1$, the difference between the two being set by radiation diffusion; therefore thermal emission is closely balanced by reabsorption, and the cooling rate is slow. By expanding (100.17) we find that as $(\kappa_0/k) \to \infty$, $n(k) \to \nu k^2/3\kappa_0^2$; therefore $t_{RR} \to 3\kappa_0^2/k^2\nu \propto (\rho c_v T_0/a_R T_0^4) \times (l^2/c\lambda_p) = (\hat{e}/E)t_d$ where $\hat{e}$ is the material energy density, E is the radiation energy density, and t_d is the radiation diffusion time.

Unno and Spiegel (**U9**) clarified the physical significance of the exact solution obtained above by using moments of the radiation field and invoking the Eddington approximation. In the limit that both hydrodynamic motions and the dynamics of the radiation field (specifically the rate of change of the radiant energy density) can be ignored, the first law of thermodynamics for the radiating fluid (96.9) yields an alternative energy equation:

$$\rho c_v(\partial T/\partial t) = -\nabla \cdot \mathbf{F}. \tag{100.18}$$

It is important to remember the restrictive assumptions on which (100.18) is based. Next, making the Eddington approximation $K_{ij} = \frac{1}{3}J\,\delta_{ij}$ and dropping the time-dependent term in the radiation momentum equation, we obtain an explicit expression for $\mathbf{F}$ [cf. (97.68)], by means of which we can rewrite (100.18) as

$$\rho c_v(\partial T/\partial t) = \nabla \cdot [(4\pi/3\kappa)\,\nabla J]. \tag{100.19}$$

The approximations made here are the same as those used in the nonequilibrium diffusion approximation (cf. §97). Then substituting for J from (100.1) we have

$$\rho c_v(\partial T/\partial t) = \nabla \cdot \{(1/3\kappa)\,\nabla[a_R c T^4 + (\rho c_v/\kappa)(\partial T/\partial t)]\}, \tag{100.20}$$

which describes the thermal behavior of a static radiating medium, in the Eddington approximation, when time evolution of the radiation field is ignored. Finally, linearizing (100.20) and using the fact that $\nabla T_0 \equiv 0$ (homogeneous medium) we find

$$(\nabla^2 - 3\kappa^2)(\partial T_1/\partial t) = -\nu\,\nabla^2 T_1, \tag{100.21}$$

where ν is defined by (100.10).

The essential physics emerges when we examine (100.21) in the opaque and transparent regimes. In opaque material $\kappa l \to \infty$, where l is a characteristic length, and (100.21) limits to the diffusion equation

$$(\partial T_1/\partial t) = (\nu/3\kappa^2)\,\nabla^2 T_1, \tag{100.22}$$

which shows that radiative relaxation occurs on a characteristic time scale $\propto 3\kappa^2 l^2/\nu$, as found above. For transparent material $\kappa l \to 0$ and (100.21) limits to *Newton's law of cooling*

$$(\partial T_1/\partial t) = -\nu T_1, \tag{100.23}$$

according to which the rate of cooling is linearly proportional to the size of the temperature fluctuation, and has a characteristic time scale $t_{RR} = 1/\nu$.

Using a trial solution of the form (100.12) in (100.21) we recover (100.13), but with the exact $n(k)$ replaced by

$$n_E(k) = \nu/[1 + 3(\kappa_0/k)^2]. \tag{100.24}$$

Equation (100.24) has the same limiting behavior as (100.17) when $(\kappa_0/k) \to 0$ and $(\kappa_0/k) \to \infty$, and, as shown in Table 100.1, provides a reasonably good approximation in between.

When the effects of scattering are included, κ_0 is replaced by $(\kappa + \sigma)_0$ in

(100.19), hence (100.24) becomes

$$n_E(k) = \nu/\{1+3[\kappa_0(\kappa+\sigma)_0/k^2]\}. \tag{100.25}$$

From (100.25) one sees that for a given total opacity $\chi_0 = (\kappa+\sigma)_0$, increasing σ_0 relative to κ_0 always increases the relaxation time. Indeed in the limit of pure scattering ($\kappa_0 = 0$) we have the noteworthy result that $n_E(k) \equiv \nu = 0$, hence the medium behaves adiabatically, as one would expect because photons are conserved by a pure scattering process. When the effects of energy exchange by thermal conduction are included (**A6**), (**D1**), the relaxation rate becomes $n = n_{\text{rad}} + n_{\text{cond}}$, where $n_{\text{cond}} = (K/\rho c_v)k^2$; here K is the material thermal conductivity. Comparison of n_{rad} and n_{cond} shows that radiation dominates only in long-wavelength disturbances, specifically when

$$k^2 < (16\sigma_R\kappa_0 T_0^3/K) - 3\kappa_0(\kappa+\sigma)_0. \tag{100.26}$$

From the fairly close agreement of $n_E(k)$ and $n(k)$ many authors have concluded that the Eddington approximation is valid for the perturbed radiation field in both the optically thick and thin limits. We can examine this conclusion critically by calculating the Eddington factor directly from (100.16), obtaining

$$\begin{aligned} f(k) &= \int_0^1 \mu^2[1+(k/\kappa_0)^2\mu^2]^{-1}\,d\mu \bigg/ \int_0^1 [1+(k/\kappa_0)^2\mu^2]^{-1}\,d\mu \\ &= (\kappa_0/k)[1-(\kappa_0/k)\cot^{-1}(\kappa_0/k)]/\cot^{-1}(\kappa_0/k). \end{aligned} \tag{100.27}$$

Numerical values for $f(k)$ are given in Table 100.1. For optically thick disturbances $f \to \frac{1}{3}$. But for optically thin disturbances f actually *vanishes*, indicating that the perturbed radiation field is far from isotropic. In fact, the perturbed radiation field has a "pancake-shaped" distribution around the normal to the plane of the disturbance, because in the plane ($\mu = 0$), $I_1(\mu) = B_1$, but as $\mu \to 1$ (i.e., along **k**) $I_1 \to 0$ because contributions from the sinusoidal variation of B sum to zero when there is no attenuation along the ray.

The real reason that $n_E = n$ when $\kappa_0/k = 0$ is *not* that the Eddington approximation is valid in this limit, but rather that $J_1 \propto (\kappa_0/k) \to 0$ in an optically thin disturbance. Hence the absorption term $\kappa_0 J_1$ in the energy equation vanishes, and the relaxation rate is set solely by the emission term $\kappa_0 B_1$, which is independent of both k and f. Indeed, retracing the derivation of (100.24), one finds that $n_E \equiv \nu$ when $\kappa_0/k = 0$ no matter what numerical value is chosen for the closure ratio K/J; that is, the Eddington approximation is *irrelevant* in the optically thin limit.

Spiegel's formula for t_{RR} has been extensively applied in estimating the effects of radiative damping on waves in stellar atmospheres (cf. §§101 and 102). But it is well to emphasize the restrictive assumptions on which it rests: an infinite, grey, homogeneous medium in LTE; initial radiative equilibrium; and no dynamics of either the matter or the radiation field. It entirely neglects *boundary* and *nonlocal transport* effects arising from inhomogeneities in the medium. The conditions under which the formula is known to be valid are therefore very limited, and one must be careful not to misapply it.

As an example of the consequences of dropping some of the assumptions made in deriving (100.17), consider an ambient homogeneous, static, LTE medium not initially in radiative equilibrium, but in a steady state under the action of radiative energy exchange and a constant nonradiative (e.g., magnetic) energy input (or loss) $\dot{q}$. Then

$$(\partial e/\partial t)_0 = 4\pi\kappa_0(J_0 - B_0) + \dot{q} = 0, \qquad (100.28)$$

which implies $J_0 \neq B_0$. (Such a state can be realized only for a *finite* homogeneous medium, for otherwise J_0 would inevitably saturate to B_0.) In this case we cannot omit the term $4\pi\kappa_1(J_0 - B_0)$ from the linearized energy equation, nor the term $\kappa_1(I_0 - B_0)$ from the linearized transfer equation. Retracing the analysis we find that the effect is to replace B_1 in both (100.6) and (100.8) by an equivalent source

$$\tilde{B}_1 = B_1 + (\partial \ln \kappa_0/\partial T)(B_0 - J_0)T_1, \qquad (100.29)$$

and therefore ν in (100.10) et seq. by

$$\tilde{\nu} = (4\pi\kappa_0/\rho c_v)[(a_R c/\pi)T_0^3 + (\partial \ln \kappa_0/\partial T)(B_0 - J_0)]. \qquad (100.30)$$

When $\dot{q} > 0$, then $B_0 > J_0$, and the nonradiative energy input is balanced by excess emission. Then if $(\partial\kappa_0/\partial T) > 0$, we have $\tilde{\nu} > \nu$, as one expects because an increase in opacity produces an increased rate of emission. If, on the other hand, $(\partial\kappa_0/\partial T) < 0$, so that the material radiates less efficiently as it is heated, then $\tilde{\nu} < \nu$, and the relaxation time increases. Indeed if the second term in (100.30) is sufficiently negative, $\tilde{\nu}$ can become negative and an initial fluctuation will grow rather than decay; in this case the material is *thermally unstable.*

TIME-DEPENDENT RADIATION TRANSPORT

To extend the analysis we now allow for the finite propagation speed of light and consider time-dependent radiation transport; we thereby allow the radiation field to have a dynamical character. As before, we assume no material motions, which means that the material will respond only passively to the radiation field. Intuitively we expect to find again a thermal relaxation mode, but modified by the finite photon flight time $t_\lambda \equiv \lambda_p/c = 1/c\kappa$, and in addition, other new modes arising from the dynamical nature of the radiation field, including attenuated propagating radiation waves that correspond to the flow of radiation through an absorbing medium.

We adjoin the radiation energy and momentum equations to the gas energy equation and for steadily driven disturbances in an infinite homogeneous medium derive a dispersion relation (which is independent of global initial-boundary conditions) for the coupled set (**A6**), (**D1**). To close the system of moments we invoke the Eddington approximation, encouraged by the good results it gives in the quasi-static case.

For an infinite, homogeneous, grey, LTE medium initially in radiative equilibrium the perturbation equations to be solved are (100.6) and

$$(1/c\kappa)(\partial J_1/\partial t)+(1/\kappa)(\partial H_1/\partial x)=B_1-J_1 \tag{100.31}$$

and

$$(1/c\kappa)(\partial H_1/\partial t)+(1/3\kappa)(\partial J_1/\partial x)=-H_1, \tag{100.32}$$

where we noted that $J_0 \equiv B_0$ and $H_0 \equiv 0$. Assuming plane-wave perturbations of the form $\phi_1 = \Phi e^{ikx}e^{-nt}$ we obtain the system

$$\begin{pmatrix} nt_\lambda - 1 & -ik/\kappa & 1 \\ -ik/3\kappa & nt_\lambda - 1 & 0 \\ \nu & 0 & n-\nu \end{pmatrix} \begin{pmatrix} J_1 \\ H_1 \\ B_1 \end{pmatrix} = 0 \tag{100.33}$$

which has a nontrivial solution only if the determinant of coefficients is zero. From this requirement we obtain the dispersion relation

$$z^3-(\alpha+2)z^2+(\alpha+\beta+1)z-\alpha\beta=0, \tag{100.34}$$

where $z \equiv nt_\lambda$, $\alpha \equiv \nu t_\lambda$, and $\beta \equiv k^2/3\kappa^2$.

In general, (100.34) has either three real roots or one real root and two conjugate complex roots. Given the roots n_i, $(i=1, 2, 3)$, one finds that the *eigenvectors* of the system have components

$$\mathbf{V}_i(k)=(J_1, H_1, B_1)_i=[1, (in_i t_\lambda \kappa/k)(n_i-\nu-t_\lambda^{-1})/(\nu-n_i), \nu/(\nu-n_i)]\times J_1. \tag{100.35}$$

To gain physical insight we examine (100.34) in various limiting cases. Suppose first that $t_\lambda \to 0$, which implies that $c\to\infty$, hence *quasi-static radiation.* We then recover (100.24) and thus have the same thermal relaxation mode as before. We find

$$J_1 = B_1/[1+(k^2/3\kappa^2)] \tag{100.36a}$$

and

$$H_1 = -(ik/3\kappa)J_1. \tag{100.36b}$$

Note that $J_1 \to B_1$ and $H_1 \to 0$ as $\tau_\Lambda \to \infty$; and $J_1 \to 0$, $H_1 \to 0$ as $\tau_\Lambda \to 0$; H_1 lags J_1 by $\pi/2$ as a function of x.

Next, suppose that $\nu \to 0$, which corresponds to material with *infinite heat capacity*. Here the state of the matter is frozen, and a disturbance in the radiating fluid can propagate only by radiation. The dispersion relation reduces to

$$z[(z-1)^2+\beta]=0, \tag{100.37}$$

which yields roots $z_1=0$ and $z_{2,3}=1\pm ik/\sqrt{3}\,\kappa$. For $n=0$ we can impose an arbitrary B_1, which does not decay in time; J_1 and H_1 are again given by

(100.36), and merely represent the static adjustment of the radiation field to an imposed source perturbation (constant in time). The other two roots are more interesting; we find $B_1 \equiv 0$ and

$$H_1 = \mp J_1/\sqrt{3} \tag{100.38a}$$

where

$$J_1 \propto e^{-t/t_\lambda} e^{ik[x \pm (c/\sqrt{3})t]}. \tag{100.38b}$$

These are *damped radiation waves* propagating along the $\pm x$ axis with a phase and group speed $c/\sqrt{3}$, attenuating in time at a given spatial position on a time scale t_λ, or in space following a particular phase crest with a spatial scale $(\sqrt{3}\,\kappa)^{-1}$. These modes were rejected by previous authors (**A6**), (**D1**), (**F5**), but are, in fact, legitimate modes in a radiating fluid. The propagation speed is $c/\sqrt{3}$ instead of c because we have used the Eddington approximation and therefore obtain the *telegrapher's equation* in the optically thin limit instead of the exact radiation wave equation [cf. the discussion following (97.110)]. To obtain a better solution we would need to calculate an accurate Eddington factor for the time-dependent radiation field.

Next consider a *homogeneous disturbance* ($k \equiv 0$). From (100.33) we find the dispersion relation

$$z(z-1)[z-(\alpha+1)] = 0 \tag{100.39}$$

which has roots $z_1 = 0$, $z_2 = 1$, and $z_3 = \alpha + 1$. For the nondecaying mode $n_1 = 0$ we find that $J_1 \equiv B_1$ and $H_1 \equiv 0$. Here we have merely reached a *new equilibrium* in the radiation field by making identical, constant changes in J and B; H_1 is zero by symmetry. For the root $n_2 t_\lambda = 1$ we find that $J_1 = B_1 = 0$, whereas H_1 is arbitrary. Thus we may apply any nonzero perturbation in the specific intensity of the form $I_1(\mu) = \Sigma a_i P_i(\mu)$ as long as $a_0 = a_2 = 0$, which implies that $J_1 = K_1 = 0$ (Eddington approximation); a_1 and a_i for $i \geq 3$ may be arbitrary. Alternatively we may impose an azimuthal anisotropy such that J_1 and K_1 are zero (**F5**). Here we have an *isotropization mode*, in which an initial angular anisotropy of the radiation field is removed by absorption and isotropic re-emission on a radiation-flow time scale t_λ; the process is analogous to the establishment of an isotropic velocity distribution function for material particles in a deflection time t_D (cf. §10). Finally, for the root $n_3 = \nu + t_\lambda^{-1}$ we find $J_1 = -B_1/\nu t_\lambda$, which yields exact energy conservation in the linearized gas energy equation, and $H_1 = 0$. Here we have an *exchange mode* in which a given amount of energy is removed from the radiation field and temporarily deposited in the material (or vice versa) thus conserving the total fluid energy, but destroying radiative equilibrium. The disturbance decays back to equilibrium at a rate even faster than the relaxation rate of a transparent disturbance because we have simultaneously increased B (hence the emissivity) and decreased J (hence the rate of absorption), or vice versa, which results in a larger temperature imbalance (hence relaxation rate) than in an optically thin disturbance where B is altered but the ambient J_0 is unperturbed.

Consider next the *opaque limit*, $\beta \ll 1$, where (100.34) yields three real roots. We find that to first order in β the smallest root of (100.34) is

$$n_1(k) \approx \nu(k^2/3\kappa^2)/(1+\nu t_\lambda), \qquad (100.40)$$

which is just the thermal relaxation mode, but with an effective relaxation rate $\nu/(1+\nu t_\lambda)$. The decreased relaxation rate is what one would expect intuitively because the radiation now takes a finite time to flow. In this mode J_1 and B_1 always have the same sign. For $\beta \ll 1$, J_1 and B_1 are nearly equal and $|H_1| \ll |J_1|$; and $J_1 \to B_1$ while $H_1 \to 0$ when $k \to 0$. The isotropization-mode root in the limit of small β is

$$n_2(k) \approx \nu\{1-[(k^2/3\kappa^2)(\nu t_\lambda - 1)/\nu t_\lambda]\}. \qquad (100.41)$$

Again we have both $|J_1| \ll |H_1|$ and $|B_1| \ll |H_1|$. Finally, the exchange-mode root in the limit of small β is

$$n_3(k) \approx \nu + t_\lambda^{-1} - [(k^2/3\kappa^2)/\nu t_\lambda^2(1+\nu t_\lambda)]. \qquad (100.42)$$

In this mode J_1 and B_1 always have opposite signs, and H_1 is small and 90° out of phase with J_1.

Finally, consider the *transparent limit* $\beta \to \infty$. We find that (100.34) has one real root

$$n_1(\infty) \approx \nu \qquad (100.43)$$

and two complex roots

$$n_{2,3}(\infty) \approx c\kappa \pm ick/\sqrt{3}. \qquad (100.44)$$

The real root corresponds to a pure damped disturbance; as we will shortly see, the mode by which it decays depends on the value of α. The complex roots correspond to two damped radiation waves propagating with phase and group speed $c/\sqrt{3}$. In these modes the sign parity of J_1 relative to B_1 is opposite to that in the surviving mode corresponding to n_1. This fact and the conjugate relation of the complex roots guarantees that with the three modes we can always synthesize an imposed perturbation in which J_1 and B_1 have arbitrary relative amplitude and phase.

The analytical results discussed above are represented in Figure 100.1, which shows $n(k)$ obtained from numerical solutions of (100.34). For optically thick disturbances we always find three real roots, corresponding to the exchange, isotropization, and thermal-relaxation modes. For small optical thickness we always find one real root and two complex roots corresponding to damped radiation waves. When $\alpha < 1$ the real root corresponds to the thermal relaxation mode; when $\alpha > 1$ it corresponds to the exchange mode. The connectivity of the various branches changes abruptly at $\alpha = 1$, as illustrated. For the special case $\alpha = 1$ the dispersion

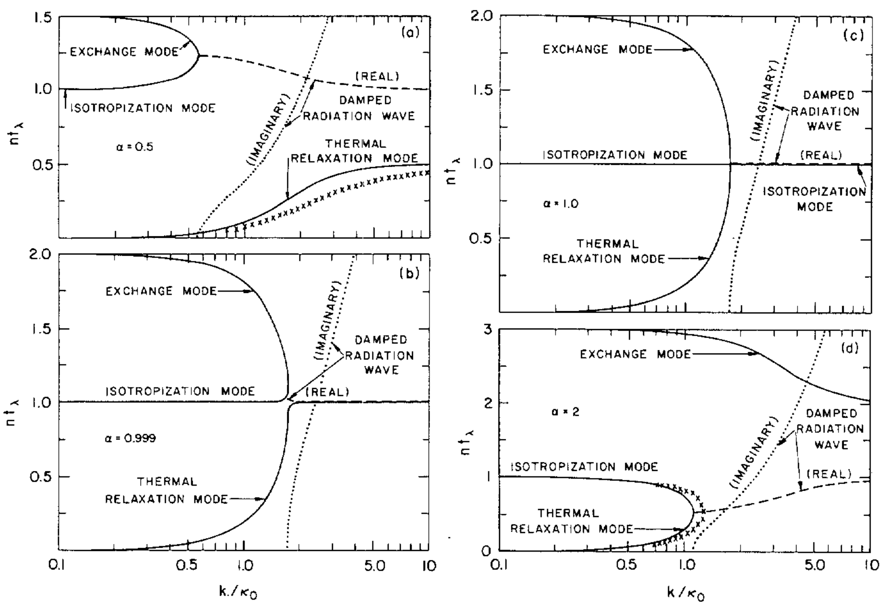

Fig. 100.1 Decay rates, as a function of optical thickness, of temperature fluctuations in a static medium.

relation

$$(z-1)(z^2-2z+\beta)=0 \qquad (100.45)$$

can be solved exactly; one finds $z_1 = 1$ and $z_{2,3} = 1 \pm (1-\beta)^{1/2}$. The first root corresponds to the isotropization mode, which survives into the transparent limit in this case; the other roots yield damped radiation waves when $\beta > 1$, and the thermal-relaxation and exchange modes when $\beta < 1$.

For a specific choice of k/κ and νt_λ, an arbitrary disturbance can be projected onto the three eigenvectors of the system, yielding three components of the perturbation. The ith component decays away on a time scale $1/\text{Re}\,(n_i)$, hence the mode with the smallest root dominates the long-term time evolution of the radiating fluid. For example, in the opaque limit, local imbalances in radiative equilibrium will be removed first by direct exchange between radiation and material energy, next the radiation field will isotropize, and finally the residual perturbation will be smoothed by thermal relaxation. For optically thin disturbances energy transport in radiation waves becomes efficient, and two of the modes that exist for optically thick disturbances will be replaced by these modes. For small α, $t_\lambda < t_{RR}$, hence the radiation field will adjust essentially instantaneously to the state of the material via damped radiation waves; the final rate of relaxation (in the thermal relaxation mode) is set by the heat capacity of the material. In contrast, for large α, $t_\lambda > t_{RR}$, hence the radiation is essentially "frozen" and the material adjusts rapidly to the *local* radiation field via the exchange mode; final global smoothing of the initial disturbance then proceeds via damped radiation waves.

All of the results discussed above are based on the Eddington approximation, which, as we saw from (100.27), breaks down for optically thin disturbances. Delache and Froeschle (**D1**), (**F5**) attempted instead to find an exact solution. They obtained a dispersion relation that yields only one or two real roots, and concluded that the complex roots of (100.34) must be rejected, thereby discarding the damped radiation waves. However, their solution encounters a severe difficulty in the optically thin regime because they find only one mode, which has the unacceptable implication that one is not free to impose an arbitrary initial disturbance, but only one with the correct relationship (both in sign and relative size) between J_1 and B_1. This lack of a complete set of modes indicates a deficiency in the analysis.

In fact, the formal solution in (**D1**) is invalid when $nt_\lambda \geq 1$ because initial-boundary conditions are not accounted for correctly [cf. (79.35)]. The mathematical symptom is that a certain integral diverges unless $nt_\lambda < 1$; physically the divergence occurs because the integral is swamped by an exponentially divergent source when the integration is extended to $t \to -\infty$ instead of being truncated at $t = 0$ (the instant when the initial perturbation was imposed). The solution in (**F5**) suffers from a similar problem. Furthermore, when correct limits are applied in the formal solution it is no longer possible to use a separable solution of the form (100.12). Thus the proposed "exact" solution appears to have only limited applicability.

In our opinion the Eddington approximation should always yield results that are at least qualitatively correct. For example, suppose $nt_\lambda < 1$. Here we expect the Eddington approximation to be valid in the opaque limit, and to become irrelevant in the transparent limit. As a test we replace the factor $\frac{1}{3}$ in (100.32) to (100.34) with the quasi-static $f(k)$ given by (100.27). As shown in Figure 100.1 we find little change in the modes with $nt_\lambda < 1$. The same remark holds for modes with $nt_\lambda \geq 1$, but in this case we cannot guarantee that (100.27) is valid. However, we know that for a mode with $nt_\lambda > 1$ the material equilibrates to the local radiation field via *isotropic* emission and absorption processes on a time scale shorter than that required for radiation to flow to (or from) adjacent regions. Therefore if we assume (legitimately) that the initial radiation perturbation is isotropic, we can argue that it must remain isotropic during the lifetime of the mode, hence that the Eddington approximation will apply. Similarly, for $nt_\lambda = 1$ the effect of the isotropization mode is to isotropize an initially anisotropic distribution. Likewise the damped radiation modes will propagate an initial isotropic disturbance isotropically. In all cases the Eddington approximation appears reasonable.

The radiative relaxation of a medium comprising non-LTE two-level atoms, radiation, and an ambient LTE gas is discussed in (**L7**), (**F5**), and (**F6**). Appropriate rate equations are adjoined to the gas energy and radiation transport equations. The resulting dispersion relation is more complicated and yields a richer spectrum of modes, which now depend on characteristic time scales governing the kinetics of statistical equilibrium (e.g., radiative and collisional rates) in addition to the parameters entering in the cases discussed above.

Virtually all astrophysical discussions of the radiative damping of temperature fluctuations and/or waves are based on (100.17) or (100.24), and thus ignore the time dependence of the radiation field. This simplification is usually justified because the dynamical time scales of many astrophysical phenomena are enormously longer than a photon flight time, hence the radiation field is indeed quasi-static and the fast exchange, isotropization, and damped-radiation-wave modes are of little interest. A similar situation is encountered in fluid dynamics where in order to follow the evolution of flow phenomena having long time scales, one can make the *anelastic approximation* and adopt modified equations of hydrodynamics that suppress sound waves, thereby filtering out variations on short time scales that otherwise are a nuisance computationally.

THERMAL RESPONSE OF THE SOLAR ATMOSPHERE

In attempting to apply (100.17) to estimate the radiative relaxation time of temperature fluctuations (or waves) in a stellar atmosphere, one must account for two important effects: (1) the variation of material properties, hence ν, with height, and (2) the presence of an open boundary.

Estimates of the relaxation time for an optically thin disturbance, $t_{RR}(\infty)$, have been made by several authors [e.g., (**B6,** 326), (**S17**), (**S24**), (**U3**)] using realistic model solar atmospheres. Allowing for continuum opacities only, one finds that $t_{RR}(\infty)$ in the photosphere ($\tau_{\text{cont}} \approx 1$) is about 1 s, and rises rapidly with height, reaching a maximum of about 800 s at about 700 km above the photosphere. At greater heights the relaxation time begins to drop because of rising temperature, then passes through a secondary maximum as hydrogen ionizes, and finally plunges sharply. These results are modified drastically when radiative losses in spectral lines are included (**G5**); one then finds that $t_{RR}(\infty)$ rises to about 500 s just above the temperature minimum, then falls to only 90 s in the mid-chromosphere where line losses are large, before rising again to about 400 s when hydrogen ionizes. Unfortunately it is difficult to allow properly for self-absorption in the spectral lines, hence to estimate accurately their net cooling rate, and the line-loss term is uncertain by at least a factor of 2.

Below $\tau_{\text{cont}} \approx 1$, $t_{RR}(\infty)$ drops rapidly as κ rises sharply. But for a disturbance of finite wavenumber k, the increase of τ_Λ with increasing κ implies that $t_{RR}(k)$ increases rapidly, in accordance with (100.17). Ultimately, $t_{RR}(k)$ becomes so large that a time-periodic disturbance behaves essentially adiabatically ($\omega t_{RR} \gg 1$). Because the atmosphere has an open boundary the effective relaxation rate of a disturbance depends on its optical depth τ in the atmosphere, as well as on τ_Λ. Thus a horizontal perturbation will relax by horizontal radiative exchange between crests and troughs if $\tau_\Lambda \ll \tau$, but if $\tau_\Lambda \gg \tau$ it relaxes more efficiently as a result of vertical radiative losses through the open boundary. Ulrich (**U8**) suggested that in calculating t_{RR} from (100.17) we use an effective optical thickness given by

$$\tau_{\text{eff}}^{-2} \equiv \tau_\Lambda^{-2} + \tau^{-2}. \tag{100.46}$$

In physical terms (100.46) gives the harmonic mean of the number of absorptions a photon requires to cross one wavelength of the disturbance, and the number to escape from the atmosphere. For a vertical disturbance one can use (100.46) or simply choose $\tau_{\text{eff}} = \min(\tau, \tau_\Lambda)$. In an exponential atmosphere $\tau_\Lambda : \tau = \Lambda : H$, hence τ_Λ sets the relaxation rate only for short-wavelength disturbances (or for long-wavelength disturbances deep in the envelope where H becomes large).

The thermal response of the solar atmosphere to periodic time variations of the radiative flux incident from below is examined numerically in (**W5**). The atmosphere is assumed to be motionless, but allowance is made for an inhomogeneous vertical structure. A sinusoidal variation with a 10 percent amplitude in the radiative flux is imposed at the lower boundary. Initial transients (the subjects of study elsewhere in this section) are allowed to die out, and the final periodic solution driven by the boundary condition is obtained. From the numerical results one finds that (1) the amplitude of the temperature fluctuation decreases with increasing height, (2) there is a phase lag between the imposed flux and the temperature response, (3) the lag increases with height, and (4) the lag is an increasing fraction of a period as the period decreases.

To understand these results qualitatively, consider the optically thin part of the atmosphere, and assume that

$$T(z, t) = T_0[1 + \xi(z)e^{i\omega t}] \tag{100.47}$$

and

$$J(t) = J_0(1 + \varepsilon e^{i\omega t}), \tag{100.48}$$

where T_0 is a suitable average and $J_0 = B_0 = \sigma_R T_0^4/\pi$. The perturbation ε is constant because the region considered is optically thin. The linearized energy equation then reduces to

$$i\omega\xi(z) = (4\sigma_R\langle\kappa\rangle T_0^3/c_v)[\varepsilon - 4\xi(z)]. \tag{100.49}$$

In general both ε and ξ are complex, but we can choose the time coordinate so that ε is real, whence we have

$$\xi_I = -\tfrac{1}{4}\nu\varepsilon\omega/(\omega^2 + \nu^2) \tag{100.50a}$$

and

$$\xi_R/\xi_I = -\nu/\omega. \tag{100.50b}$$

Therefore

$$|\xi(z)| = \tfrac{1}{4}\nu(z)\varepsilon/[\omega^2+\nu^2(z)]^{1/2} \qquad (100.51a)$$

and

$$\phi(z) = -\tan^{-1}[\omega/\nu(z)]. \qquad (100.51b)$$

Here ϕ is the phase angle between J_1 and T_1; negative ϕ indicates that T lags J. In the high-frequency limit, $\omega/\nu \to \infty$,

$$|\xi| \to \tfrac{1}{4}\nu\varepsilon/\omega \to 0 \qquad (100.52a)$$

and

$$\phi \to -\frac{\pi}{2}, \qquad (100.52b)$$

so the thermal response lags the input by 90° and its amplitude vanishes. In the low-frequency limit, $\omega/\nu \to 0$,

$$|\xi| \to \tfrac{1}{4}\varepsilon \qquad (100.53a)$$

and

$$\phi \to 0, \qquad (100.53b)$$

hence the atmosphere passes through a series of quasi-equilibria with vanishing phase lag. Equation (100.51a) shows that the phase lag must increase with height because ν decreases outward through the photosphere; the approximate results given by (100.51) are in good agreement with the numerical results.

The radiative relaxation of a two-dimensional checkerboard distribution of temperature and density fluctuations simulating a hydrodynamic model of convection cells in the solar atmosphere is discussed in (**L8**). The relaxation rate is found to depend on the cell size, the size of the velocity field, and the amplitude of the initial temperature fluctuation.

101. Propagation of Acoustic Waves in a Radiating Fluid

In this section we examine the effects of radiation on acoustic waves propagating in an infinite, homogeneous medium in LTE, initially in radiative equilibrium. We first consider wave damping by radiative energy exchange, which is generally very efficient, especially near boundary surfaces (e.g., the solar photosphere) where radiative relaxation times are very short compared to typical wave periods. By comparison, wave damping by viscosity and thermal conduction is negligible in most situations of astrophysical interest. We treat the spatial damping of driven harmonic disturbances (i.e., real ω and complex k) as opposed to the time decay of a transient initial disturbance (i.e., real k and complex ω). We then consider the more fundamental role played by radiation through its contributions to the total energy density and pressure in the fluid; we find that radiation can radically alter the dynamical properties of wave modes in the fluid. Finally, we consider briefly the effects of radiation forces on acoustic waves.

NEWTONIAN COOLING (OPTICALLY THIN PERTURBATIONS)

Consider first an optically thin disturbance in which radiative energy exchange is adequately described by the Newtonian cooling approximation (**S16**), (**S25**). We assume that the radiation field is quasi-static and ignore the dynamical behavior of the radiation component of the radiating fluid. From (100.23) the net heat input to the gas is then

$$(Dq/Dt) = -\rho c_v \nu T_1 \qquad (101.1)$$

where ν is given by (100.10). Using (101.1) in (52.19) we can write the gas energy equation as

$$(Dp/Dt) - a^2(D\rho/Dt) = -16(\Gamma_3 - 1)\sigma_R \kappa_0 T_0^4 (T_1/T_0). \qquad (101.2)$$

For the special case of a perfect gas with constant specific heats, we can rewrite the right-hand side of (101.2) in a more convenient form. For such a gas $\Gamma_3 = \gamma$, $(\gamma-1)c_v = R$, $a^2 = \gamma p/\rho$, and $T_1/T_0 = (p_1/p_0) - (\rho_1/\rho_0)$, and the linearized version of (101.2) reduces to

$$(\partial p_1/\partial t) = a^2(\partial \rho_1/\partial t) + \mathbf{v}\cdot(a^2\,\nabla\rho_0 - \nabla p_0) - (a^2\rho_0/\gamma t_{RR})[(p_1/p_0) - (\rho_1/\rho_0)], \qquad (101.3)$$

where for brevity we write $t_{RR} \equiv t_{RR}(\infty) = \nu^{-1}$. Note in passing that for an ionizing gas the linearized energy equation is

$$(\partial p_1/\partial t) = a^2(\partial \rho_1/\partial t) + \mathbf{v}\cdot(a^2\,\nabla\rho_0 - \nabla p_0) - a^2\rho_0[\alpha_2(p_1/p_0) - \alpha_1(\rho_1/\rho_0)], \qquad (101.4)$$

where, from (54.84a), α_1 and α_2 for a pure hydrogen gas are

$$\alpha_1 \equiv [16(\Gamma_3 - 1)\sigma_R\kappa_0 T_0^4/\Gamma_1 p_0]/\{1 + \tfrac{1}{2}x(1-x)[\tfrac{5}{2} + (\varepsilon_I/kT)]\} \qquad (101.5)$$

and

$$\alpha_2 \equiv [1 + \tfrac{1}{2}x(1-x)]\alpha_1. \qquad (101.6)$$

For a homogeneous medium the gradient terms in (101.3) vanish, hence the dynamics of a radiatively damped acoustic wave is determined (in the Newtonian cooling approximation) by

$$(\partial \rho_1/\partial t) = -\rho_0\,\nabla\cdot\mathbf{v}_1, \qquad (101.7)$$

$$\rho_0(\partial \mathbf{v}_1/\partial t) = -\nabla p_1, \qquad (101.8)$$

and

$$(\partial p_1/\partial t) - a^2(\partial \rho_1/\partial t) = -(a^2\rho_0/\gamma t_{RR})[(p_1/p_0) - (\rho_1/\rho_0)]. \qquad (101.9)$$

Taking $(\partial^2/\partial t^2)$ of (101.9) and using (48.6), which follows from (101.7) and

(101.8), we obtain the wave equation

$$\{[(\partial^2/\partial t^2) - a^2\nabla^2](\partial/\partial t) + t_{RR}^{-1}[(\partial^2/\partial t^2) - (a^2/\gamma)\nabla^2]\}p_1 = 0. \quad (101.10)$$

For plane waves (101.10) yields the dispersion relation

$$k^2 = \left(\frac{\gamma\omega^2}{a^2}\right)\left[\frac{(1+\gamma\omega^2 t_{RR}^2) - i(\gamma-1)\omega t_{RR}}{1+\gamma^2\omega^2 t_{RR}^2}\right]. \quad (101.11)$$

Because k is complex we find damped progressive waves varying as $e^{i(\omega t - k_R x)}e^{-k_I x}$.

In the high-frequency limit, $\omega t_{RR} \gg 1$, and

$$k \approx (\omega/a)\{1 - i[\tfrac{1}{2}(\gamma-1)/\gamma\omega t_{RR}]\}, \quad (101.12)$$

which corresponds to acoustic waves traveling with the adiabatic sound speed, having a characteristic damping length

$$L \approx [2\gamma/(\gamma-1)]at_{RR}. \quad (101.13)$$

Note that $L/\Lambda \sim \omega t_{RR} \gg 1$, hence in the limit of high frequencies and/or long damping times, acoustic waves behave essentially adiabatically and suffer only a small damping *per cycle*. Note, however, that (101.13) shows that the geometrical distance over which the wave damps can be made arbitrarily small by making the relaxation time sufficiently short; in this sense high-frequency waves can be heavily damped.

In the low-frequency limit, $\omega t_{RR} \ll 1$, and

$$k \approx (\omega/a_T)[1 - i\tfrac{1}{2}(\gamma-1)\omega t_{RR}], \quad (101.14)$$

where a_T is the isothermal sound speed $a/\gamma^{1/2}$. We now have acoustic waves traveling with speed a_T, with a damping length

$$L \approx [2/(\gamma-1)]a_T t_{RR}/(\omega t_{RR})^2, \quad (101.15)$$

which implies that $L/\Lambda \sim (\omega t_{RR})^{-1} \gg 1$. Thus according to Newtonian cooling theory, in the limit of low frequencies and/or short radiative relaxation times radiative exchange obliterates temperature fluctuations in the gas, and acoustic waves propagate isothermally with negligible spatial damping.

For arbitrary ωt_{RR} one solves (101.10) numerically. As shown in Figure 101.1, the phase speed $v_p = \omega/k_R$ rises abruptly from a_T to a near $\omega t_{RR} \approx 1$. At the same time, the damping length $L/\Lambda = |k_R/2\pi k_I|$ passes through a minimum. Indeed, near $\omega t_{RR} \sim 1$, $L/\Lambda \approx 1$, so these waves decay after traveling only a few wavelengths.

It must be emphasized that all of the above results apply only for optically thin disturbances, $\kappa/k \ll 1$, $\Lambda/\lambda_p \ll 1$. The significance of this remark will become clear shortly.

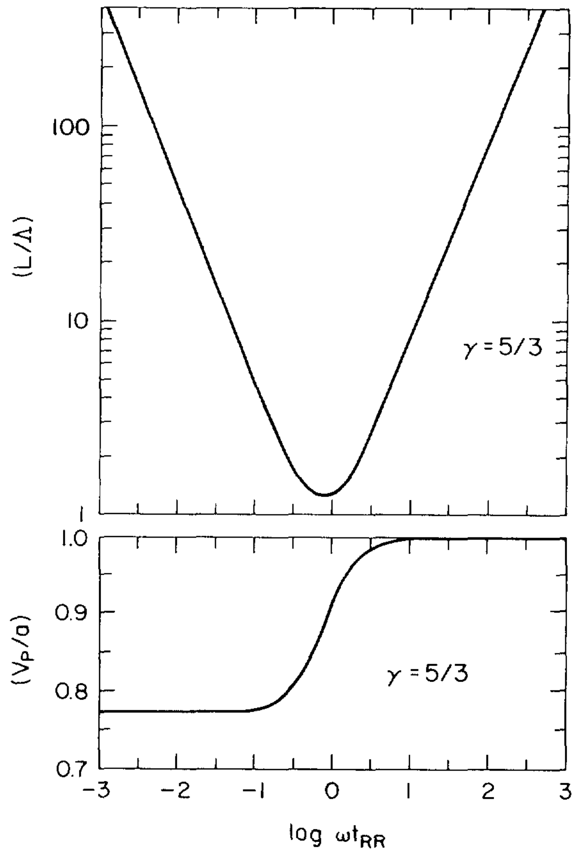

Fig. 101.1 Damping length and phase speed of acoustic mode in Newtonian cooling approximation.

EQUILIBRIUM DIFFUSION (OPTICALLY THICK DISTURBANCES)

In extremely opaque material (e.g., inside a star), radiation comes into thermal equilibrium with the matter, and energy exchange proceeds by diffusion; in this regime radiation can have important dynamical effects on wave propagation. In treating the radiation field we can apply the equilibrium diffusion approximation provided that the disturbance is sufficiently optically thick, that is, that $\kappa/k \gg 1$, and $\Lambda/\lambda_p \gg 1$.

The dynamical equations for a radiating fluid in the equilibrium diffusion regime were derived in §97. From (97.5) and (97.6) the momentum equation is

$$\rho(D\mathbf{v}/Dt) = -\nabla\bar{p} \qquad (101.16)$$

and the energy equation is

$$\rho\{(D\bar{e}/Dt) + \bar{p}[D(1/\rho)/Dt]\} = \nabla \cdot (\bar{K}\,\nabla T). \qquad (101.17)$$

Here $\bar{p}$, $\bar{e}$, and $\bar{K}$ are the *total* pressure, energy density, and conductivity of the fluid:

$$\bar{p} = p_{\text{gas}} + p_{\text{rad}} = \rho RT + \tfrac{1}{3}a_R T^4 \equiv (1+\alpha)\rho RT, \qquad (101.18)$$

$$\bar{e} = e_{\text{gas}} + (a_R T^4/\rho) = \tfrac{3}{2}RT + (a_R T^4/\rho), \qquad (101.19)$$

and

$$\bar{K} = K_{\text{thermal}} + K_{\text{rad}}, \qquad (101.20)$$

where K_{rad} is defined by (97.3). Using (52.19) we rewrite (101.17) as

$$(D\bar{p}/Dt) - a^2(D\rho/Dt) = (\Gamma_3 - 1)\,\boldsymbol{\nabla}\cdot(\bar{K}\,\boldsymbol{\nabla}T) \qquad (101.21)$$

where

$$a^2 \equiv \Gamma_1 \bar{p}/\rho = (1+\alpha)\Gamma_1(a_{\text{gas}}/\gamma), \qquad (101.22)$$

and Γ_3 and Γ_1 are given by (70.18) and (70.22).

For a small disturbance we linearize these equations. The linearized continuity and momentum equations are the same as (101.7) and (101.8) (with p_1 replaced by $\bar{p}_1$), and therefore again yield (48.6). The linearized energy equation is

$$(\partial\bar{p}_1/\partial t) - a^2(\partial\rho_1/\partial t) = (\Gamma_3 - 1)\bar{K}\,\nabla^2 T_1. \qquad (101.23)$$

We can eliminate T_1 in favor of $\bar{p}_1$ and ρ_1 by means of the linearized equation of state

$$T_1/T_0 = [(1+\alpha)(\bar{p}_1/\bar{p}_0) - (\rho_1/\rho_0)]/(1+4\alpha). \qquad (101.24)$$

Thus using (101.24) in (101.23) and making use of (70.16), (70.18), and (70.22) we find

$$(\partial\bar{p}_1/\partial t) - a^2(\partial\rho_1/\partial t) = \Gamma\chi[\nabla^2\bar{p}_1 - (a^2/\Gamma)\,\nabla^2\rho_1] \qquad (101.25)$$

where we defined an effective Γ as

$$\Gamma \equiv (1+\alpha)\Gamma_1 \qquad (101.26)$$

and the thermal diffusivity is

$$\chi \equiv \bar{K}/\rho_o c_p. \qquad (101.27)$$

With these definitions (101.25) is formally identical to (51.5) for a heat-conducting gas. Taking $(\partial^2/\partial t^2)$ of (101.25) and using (48.6) to eliminate ρ_1 we obtain the wave equation

$$\{[(\partial^2/\partial t^2) - a^2\,\nabla^2](\partial/\partial t) - \Gamma\chi[(\partial^2/\partial t^2) - (a^2/\Gamma)\,\nabla^2]\,\nabla^2\}\bar{p}_1 = 0. \qquad (101.28)$$

For a plane wave (101.28) yields the dispersion relation

$$(ak/\omega)^4 - [\Gamma - i(a^2/\chi\omega)](ak/\omega)^2 - i(a^2/\chi\omega) = 0, \qquad (101.29)$$

which is quadratic in $(ak/\omega)^2$, and contains two dimensionless numbers: Γ and $\chi\omega/a^2$. This dispersion relation is formally identical to (51.16) for a thermally conducting inviscid material gas. Hence we obtain the same modes and physical interpretation as before: an adiabatic (isothermal) radiation-modified acoustic wave, and a slow (fast) radiation-diffusion wave in the low (high) frequency limits, respectively. These waves have the same propagation characteristics as described in §51 (cf. Figures 51.1 and 51.2); only the thermodynamic parameters (χ, Γ, a) differ from their earlier definitions. Thus in the equilibrium diffusion regime radiation is an inseparable part of the radiating fluid, with photons behaving dynamically like "honorary material particles".*

It should be noted that as $\alpha \to \infty$, Γ, a, and c_p diverge, but this is an artifact of having ignored the rest energy of the fluid in our analysis [cf. (48.32) and (70.27) for a]. The divergence occurs only at extremely high temperatures where relativistic effects are major and a relativistic analysis is required. In contrast, it follows from (101.22) and (101.26) that $a^2/\Gamma \equiv (a_T^2)_{\text{gas}}$, hence the phase speed of the isothermal acoustic wave depends on gas properties only.

The analysis can be extended to include the effects of electromagnetic fields in an ionized plasma; see (**P1,** §8.2).

TIME-INDEPENDENT TRANSPORT (EDDINGTON APPROXIMATION)

The Newtonian cooling and equilibrium diffusion approximations conflict with each other in that they predict opposite variations of the propagation speed (i.e., adiabatic versus isothermal) of the acoustic mode in going from low to high frequency. This contradiction arises because each scheme breaks down in one or the other limit. Thus at a sufficiently low frequency the wavelength of a disturbance is so long that it becomes optically thick (no matter how transparent the material), and the Newtonian cooling approximation no longer applies. Conversely, at very high frequencies the wavelength of a disturbance becomes so small that it is optically thin (no matter how opaque the material) and the diffusion approximation is no longer valid because a photon mean free path exceeds the characteristic spatial scale of gradients in the disturbance (recall the discussion of flux limiting in §97).

These considerations show that it is imperative to account for *transport effects* arising from finite photon mean free paths in the disturbance. Our qualitative expectation based on the diffusion approximation is that waves should be adiabatic at very low frequencies, and become isothermal above some critical frequency; but then at some sufficiently high frequency the waves should again become adiabatic, as predicted by the Newtonian cooling approximation. Precisely this behavior was found by Stein and Spiegel (**S23**) in their analysis of the time decay of an initial disturbance, allowing for transport effects (but ignoring the time dependence and dynamical behavior of the radiation field). In keeping with the rest of the

* We are indebted to Dr. J. I. Castor for this felicitous expression.

discussion of this section, we consider instead the spatial damping of a driven disturbance along the lines explored by Vincenti and Baldwin (**V6**), (**V7,** §§12.5–12.8).

The dynamical behavior of the material is governed by the continuity equation, the material momentum equation

$$\rho(D\mathbf{v}/Dt) = -\nabla p, \tag{101.30}$$

and the gas energy equation

$$(Dp/Dt) - a^2(D\rho/Dt) = 4\pi(\gamma - 1)\kappa(J - B). \tag{101.31}$$

Here p refers to the gas pressure only, and $a^2 = \gamma p/\rho$. In (101.30) we have neglected the radiation force on the material.

We assume that the radiation field is quasi-static and make the Eddington approximation. The radiation energy equation is then

$$(\partial H/\partial x) = \kappa(B - J) \tag{101.32}$$

and the radiation momentum equation is

$$\tfrac{1}{3}(\partial J/\partial x) = -\kappa H. \tag{101.33}$$

Equations (101.32) and (101.33) provide a significant improvement over the Newtonian cooling approximation because they apply in both the optically thick and thin limits. They are also an improvement over equilibrium diffusion because they discriminate between J and B, which is crucial when the disturbance becomes optically thin. However they sacrifice some of the logical consistency inherent in the equilibrium diffusion analysis because they ignore the dynamics of the radiation field; we will remedy that flaw later.

In linearizing the radiation equations we note that $J_0 \equiv B_0$ and $H_0 \equiv 0$, and introduce nondimensional radiation variables $j_1 \equiv J_1/B_0$, $h_1 \equiv H_1/B_0$, and $B_1/B_0 = 4T_1/T_0 \equiv 4\theta_1$. Combining the linearized forms of (101.32) and (101.33) we have

$$(3\kappa^2)^{-1}(\partial^2 j_1/\partial x^2) = j_1 - 4\theta_1. \tag{101.34}$$

In linearizing the continuity and material momentum and energy equations we use a velocity potential $u_1 \equiv (\partial\phi_1/\partial x)$ which implies that $(\partial\rho_1/\partial t) = -\rho_0(\partial^2\phi_1/\partial x^2)$ and, from (101.30), $p_1 = -\rho_0(\partial\phi_1/\partial t)$. Using these expressions in the linearized gas energy equation we find

$$(\partial^2\phi_1/\partial t^2) - a^2(\partial^2\phi_1/\partial x^2) = (4a^3\kappa/\mathrm{Bo})(4\theta_1 - j_1) \tag{101.35}$$

where Bo is the Boltzmann number obtained by setting the characteristic flow speed equal to the sound speed:

$$\mathrm{Bo} \equiv \rho_0 c_p a/\sigma_R T_0^3. \tag{101.36}$$

Similarly the linearized equation of state for the material can be written

$$(\partial^2\phi_1/\partial t^2) - (a^2/\gamma)(\partial^2\phi_1/\partial x^2) + (a^2/\gamma)(\partial\theta_1/\partial t) = 0. \tag{101.37}$$

For a plane wave (101.34), (101.35), and (101.37) imply

$$\begin{pmatrix} 0 & -4 & 1+(k^2/3\kappa^2) \\ (a^2k^2/\omega^2)-\gamma & ia^2/\omega & 0 \\ (a^2k^2/\omega^2)-1 & -16a^3\kappa/\omega^2\text{Bo} & 4a^3\kappa/\omega^2\text{Bo} \end{pmatrix} \begin{pmatrix} \phi_1 \\ \theta_1 \\ j_1 \end{pmatrix} = 0. \quad (101.38)$$

Setting the determinant of (101.38) equal to zero we obtain the dispersion relation

$$[1-i(16\tau_a/\text{Bo})](ak/\omega)^4-[1-3\tau_a^2-i(16\gamma\tau_a/\text{Bo})](ak/\omega)^2-3\tau_a^2=0. \quad (101.39)$$

Here

$$\tau_a \equiv a\kappa/\omega = \tau_\Lambda/2\pi, \quad (101.40)$$

where τ_Λ is the optical thickness of one wavelength of a disturbance of frequency ω traveling with the adiabatic sound speed a. Like (101.29), (101.39) is quadratic in (a^2k^2/ω^2), hence we again get two distinct wave modes. One is a radiation-modified acoustic wave; the other is a nonequilibrium radiation diffusion wave analogous to a thermal wave.

The importance of radiation to the behavior of these waves is measured by Bo. In the limit $\text{Bo} \to \infty$ radiative energy exchange with the material ceases. In this special case the dispersion relation factors into

$$[(ak/\omega)^2-1][(ak/\omega)^2+3\tau_a^2]=0, \quad (101.41)$$

and we obtain (1) an undamped adiabatic acoustic wave in which θ_1 and v_1 are related by (48.24b), and J_1 and B_1 are related by (100.36), and (2) a radiation-field perturbation j_1 decaying as $\exp(-\sqrt{3}\,\kappa x)$ [as one expects for the Eddington approximation, cf. (83.7)], while $\phi_1=\theta_1\equiv 0$.

Solving (101.39) for $\tau_a \ll 1$ (small optical thickness and/or high frequency) we find a weakly damped acoustic wave with

$$k \approx (\omega/a)[1-i8\tau_a(\gamma-1)/\text{Bo}], \quad (101.42)$$

which implies that $v_p=a$ and $L/\Lambda=[\text{Bo}/16\pi\tau_a(\gamma-1)] \gg 1$, and a fast, strongly damped radiation diffusion wave with

$$k \approx (\omega/a)[(8\sqrt{3}\,\gamma\tau_a^2/\text{Bo})-i\sqrt{3}\,\tau_a], \quad (101.43)$$

which implies that $v_p/a=\text{Bo}/8\sqrt{3}\,\gamma\tau_a^2$ and $L=1/\sqrt{3}\,\kappa$ or $L/\Lambda=4\gamma\tau_a/\pi\text{Bo}$. Note that as $\tau_a \to 0$, the damping length for the acoustic mode becomes infinite, whereas the diffusion mode has an infinite phase speed and a fixed geometrical damping length, while $L/\Lambda \to 0$. The infinite propagation speed of the diffusion mode reflects the failure of the quasi-static radiation equations to provide flux limiting in optically thin material.

For $\tau_a \gg 1$ (i.e., large optical thickness and/or low frequency) we find a damped acoustic wave with

$$k \approx (\omega/a)[1-i8(\gamma-1)/3\tau_a\text{Bo}], \quad (101.44)$$

which implies that $v_p = a$ and $L/\Lambda = [3\tau_a \mathrm{Bo}/16\pi(\gamma-1)] \gg 1$, and a slow, heavily damped diffusion wave with

$$k \approx (\omega/a)(3\tau_a \mathrm{Bo}/32)^{1/2}(1-i), \tag{101.45}$$

which implies that $v_p/a = (32/3\tau_a \mathrm{Bo})^{1/2} \ll 1$ and $L/\Lambda = 1/2\pi$. The prediction by (101.44) that the acoustic-mode speed always equals the material sound speed independent of the radiation energy density (i.e., Bo) is in contradiction with equilibrium diffusion theory (which is valid when $\tau_a \gg 1$) and reflects the failure of (101.32) and (101.33) to account for the dynamics of the radiation field.

In general, (101.39) must be solved numerically. Results for various values of Bo are shown in Figures 101.2 and 101.3. For context, Bo is of order 10 in the solar photosphere and at the Sun's center, 10^{-2} at the center of an O-star, 10^{-5} in an X-ray source, and 10^{-14} or smaller in a solar flare. Figure 101.2 shows that the acoustic mode is indeed adiabatic at high and low frequencies, and is isothermal over a range approximately inversely proportional to Bo. Furthermore, we see that the damping length is large when the phase speed is constant, but drops sharply where v_p makes a transition between a and a_T. Figure 101.3 shows that v_p in the

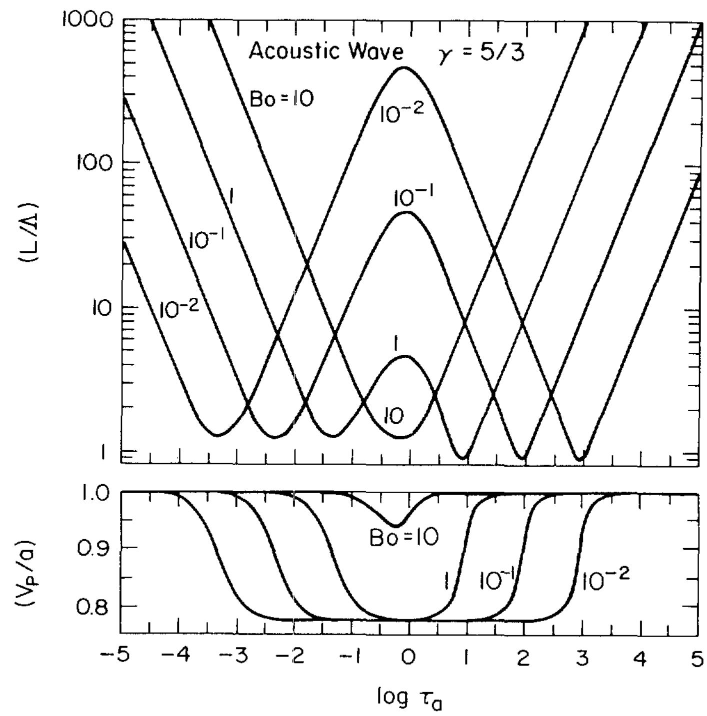

Fig. 101.2 Damping length and phase speed of acoustic mode for quasi-static radiation field, allowing for transport effects.

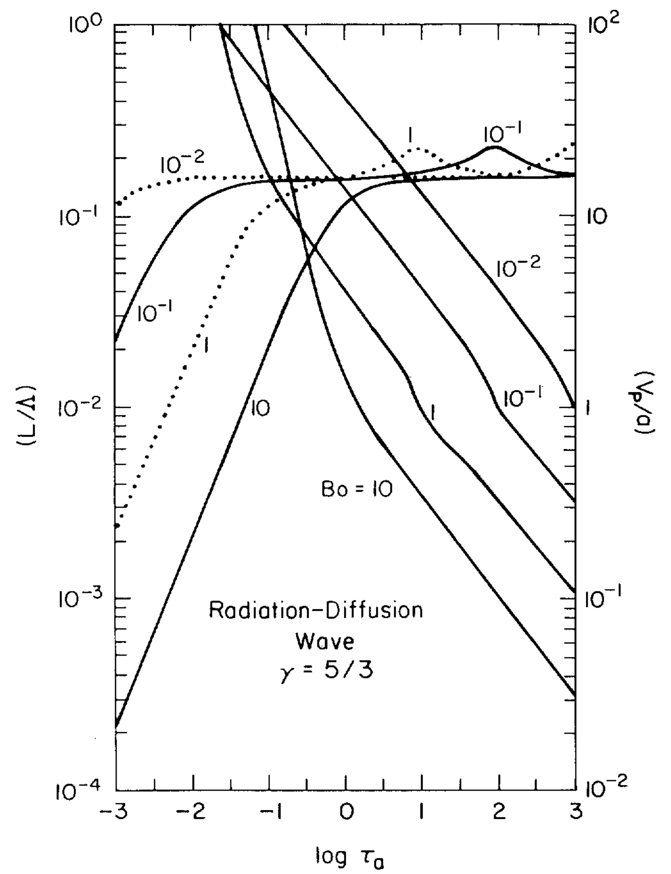

Fig. 101.3 Damping length and phase speed of radiation diffusion mode for quasi-static radiation field, allowing for transport effects.

radiation diffusion wave is always a decreasing function of τ_a, and increases with Bo when τ_a is small, but decreases with increasing Bo when τ_a is large. Near the low-frequency minimum of L/Λ for the acoustic mode, L/Λ for the radiation-diffusion mode has a local maximum like that of thermal waves as shown in Figure 51.2. At high frequencies, L is fixed but L/Λ decreases because Λ increases [because v_p increases as ω^2, see (101.43)].

TIME-DEPENDENT TRANSPORT (EDDINGTON APPROXIMATION)

The two main defects of the analysis just presented are that (1) the time dependence of the radiation field is ignored, hence propagating radiation waves are spuriously suppressed and the radiation diffusion wave is not flux limited, and (2) the dynamical effects of the radiation (work done by radiation pressure and the rate of change of the radiation energy density in the radiating fluid) are neglected. Therefore, to complete the physical picture we account for these phenomena by including the radiation pressure gradient in the material momentum equation

$$\rho(D\mathbf{v}/Dt) = -\nabla p - \nabla P \tag{101.46}$$

and by using the Lagrangean radiation energy and momentum equations

$$c^{-1}(DJ/Dt) - (4J/3c\rho)(D\rho/Dt) + (\partial H/\partial x) = \kappa(B - J) \tag{101.47}$$

and

$$c^{-1}(DH/Dt) + \tfrac{1}{3}(\partial J/\partial x) = -\kappa H. \tag{101.48}$$

The gas-energy equation (101.31) remains the same as before.

In (101.47) and (101.48), we have made the Eddington approximation, so in (101.46) $P = \frac{1}{3}E = (4\pi/3c)J$. In (101.46) we have neglected the time derivative of H, which is permissible because that term is at most $O(a/c)$ relative to ∇P, which in turn produces terms that are only $O(a/c)$ relative to the dominant terms in the dispersion relation (except at very small Boltzmann numbers).

The linearized continuity equation again yields $(\partial\rho_1/\partial t) = -\rho_0(\partial^2\phi_1/\partial x^2)$, while the linearized material momentum equation is

$$\rho_0(\partial u_1/\partial t) = -(\partial p_1/\partial x) - (4\pi B_0/3c)(\partial j_1/\partial x), \tag{101.49}$$

which implies

$$p_1 = -\rho_0(\partial\phi_1/\partial t) - (4\pi B_0/3c)j_1. \tag{101.50}$$

Using these expressions in the linearized gas-energy equation and material equation of state we find

$$(\partial^2\phi_1/\partial t^2) - a^2(\partial^2\phi_1/\partial x^2) + [4a^3/3c(\gamma-1)\mathrm{Bo}](\partial j_1/\partial t) = (4a^3\kappa/\mathrm{Bo})(4\theta_1 - j_1) \tag{101.51}$$

and

$$(\partial^2\phi_1/\partial t^2) - (a^2/\gamma)(\partial^2\phi_1/\partial x^2) + (a^2/\gamma)(\partial\theta_1/\partial t) + [4a^3/3c(\gamma-1)\mathrm{Bo}](\partial j_1/\partial t) = 0. \tag{101.52}$$

For a plane wave, (101.51) and (101.52) become

$$(a^2k^2 - \omega^2)\phi_1 - (16a^3\kappa/\mathrm{Bo})\theta_1 + (4a^3\kappa/\mathrm{Bo})[1 + i\tfrac{1}{3}(\gamma-1)^{-1}\tau_c^{-1}]j_1 = 0, \tag{101.53}$$

and

$$(a^2k^2 - \gamma\omega^2)\phi_1 + ia^2\omega\theta_1 + (4a^3\kappa/\mathrm{Bo})[i\tfrac{1}{3}\gamma(\gamma-1)^{-1}\tau_c^{-1}]j_1 = 0. \tag{101.54}$$

By analogy with (101.40) we have defined

$$\tau_c \equiv c\kappa/\omega, \tag{101.55}$$

the optical thickness associated with a disturbance of frequency ω traveling at the speed of light (not sound). Notice that $\tau_c : \tau_a = c : a$, hence in an acoustic wave of any appreciable optical thickness $\tau_c \gg 1$.

The linearized radiation equations are

$$(c\kappa)^{-1}(\partial j_1/\partial t)+(4/3c\kappa)(\partial^2\phi_1/\partial x^2)+\kappa^{-1}(\partial h_1/\partial x)=4\theta_1-j_1 \quad (101.56)$$

and

$$(c\kappa)^{-1}(\partial h_1/\partial t)+(3\kappa)^{-1}(\partial j_1/\partial x)=-h_1, \quad (101.57)$$

where we noted that $J_0\equiv B_0$ and $H_0\equiv 0$. For plane waves, (101.56) and (101.57) become

$$-(4k^2/3c\kappa)\phi_1-4\theta_1+(1+i\tau_c^{-1})j_1-i(k/\kappa)h_1=0 \quad (101.58)$$

and

$$-i(k/3\kappa)j_1+(1+i\tau_c^{-1})h_1=0, \quad (101.59)$$

which, when combined, yield

$$-(4k^2/3c\kappa)(1+i\tau_c^{-1})\phi_1-4(1+i\tau_c^{-1})\theta_1+[(1+i\tau_c^{-1})^2+(k^2/3\kappa^2)]j_1=0. \quad (101.60)$$

Thus we have

$$\begin{pmatrix} -(4k^2/3c\kappa)(1+i\tau_c^{-1}) & -4(1+i\tau_c^{-1}) & (1+i\tau_c^{-1})^2+(k^2/3\kappa^2) \\ (a^2k^2/\omega^2)-\gamma & ia^2/\omega & i(4a^3\kappa/\omega^2\mathrm{Bo})\tfrac{1}{3}\gamma(\gamma-1)^{-1}\tau_c^{-1} \\ (a^2k^2/\omega^2)-1 & -16a^3\kappa/\omega^2\mathrm{Bo} & (4a^3\kappa/\omega^2\mathrm{Bo})[1+i\tfrac{1}{3}(\gamma-1)^{-1}\tau_c^{-1}] \end{pmatrix} \times \begin{pmatrix} \phi_1 \\ \theta_1 \\ j_1 \end{pmatrix}=0. \quad (101.61)$$

From the determinant of (101.61) we obtain, after some reduction, the dispersion relation

$$\begin{aligned} &[1-i(16\tau_a/\mathrm{Bo})]z^4 \\ &\quad+\{3\tau_a^2(1+i\tau_c^{-1})^2-1+i(16\gamma\tau_a/\mathrm{Bo})+(16a/c\mathrm{Bo})\tau_a^2(1+i\tau_c^{-1}) \\ &\quad\times[5+i\tfrac{1}{3}(\gamma-1)^{-1}\tau_c^{-1}+(16a/3c\mathrm{Bo})\gamma(\gamma-1)^{-1}]\}z^2 \\ &\quad-3\tau_a^2[(1+i\tau_c^{-1})^2+(16\gamma a/c\mathrm{Bo})(1+i\tau_c^{-1})]=0, \end{aligned} \quad (101.62)$$

where $z\equiv ak/\omega$. Equation (101.62) is more complicated than (101.39) and admits a richer variety of wave modes. It is easy to study analytically only in limiting cases. Notice that (101.62) contains yet another dimensionless parameter $r\equiv a/c\mathrm{Bo}$; we consider the cases of small and large r separately.

In most laboratory experiments and familiar stellar astrophysical regimes, temperatures are low enough to guarantee that $r\ll 1$ because $a/c\ll 1$, even though Bo may be much smaller than unity and radiation makes a significant contribution to the energy-momentum balance in the fluid. For example, at the center of the Sun $\mathrm{Bo}\sim 10$, $a/c\sim 10^{-3}$, hence $r\sim 10^{-4}$; at the center of an O-star $\mathrm{Bo}\sim 10^{-2}$, $a/c\sim 2\times 10^{-3}$, hence $r\sim 0.2$.

In the small-r regime we drop terms in r and r^2 from (101.62), and analyze

$$[1-i(16\tau_a/\text{Bo})]z^4+[3\tau_a^2(1+i\tau_c^{-1})^2-1+i(16\gamma\tau_a/\text{Bo})]z^2-3\tau_a^2(1+i\tau_c^{-1})^2=0. \tag{101.63}$$

It is evident that for large τ_a, (101.63) reduces to (101.39) because $\tau_c \gg \tau_a$. Hence for $\tau_a \gg 1$ the behavior of the modes is essentially the same as discussed above for quasi-static radiation. On the other hand, for $\tau_a \ll \tau_c \lesssim 1$, the time dependence of the radiation field becomes important. In this limit (101.63) factors approximately into

$$(z^2-1)[z^2+3\tau_a^2(1+i\tau_c^{-1})^2]\approx 0. \tag{101.64}$$

Equation (101.64) has two roots: $k \approx \omega/a$, corresponding (formally) to an adiabatic acoustic wave, and

$$k \approx \sqrt{3}\,[(\omega/c)-i\kappa], \tag{101.65}$$

corresponding to a damped radiation wave propagating with speed $c/\sqrt{3}$ (Eddington approximation). This (flux-limited) radiation wave displaces the radiation diffusion wave at moderate-to-small values of τ_c. The geometrical damping length of this mode remains fixed at $L=1/\sqrt{3}\,\kappa$, whereas $L/\Lambda=(1/2\pi\tau_c)\to\infty$ as $\tau_c\to 0$. The acoustic mode is also damped; analysis of (101.63) shows that to first order in τ_a we recover (101.42). This result is, of course, only formal, as an acoustic wave cannot exist at frequencies characteristic of light waves because internal processes in the gas invalidate the inviscid continuum description of the fluid at much lower frequencies.

As the temperature of the fluid is raised, (a/c) increases and Bo decreases. Thus the ratio r may eventually become of order unity or greater; for example, in an X-ray source $(a/c)\sim 2\times 10^{-3}$ while $\text{Bo}\sim 10^{-5}$, hence $r\sim 200$. In this regime we must therefore analyze the full dispersion relation (101.62). The analysis shows that for $\tau_a \ll 1$ we recover (101.42) and (101.65), so we again have an attenuated radiation wave and (formally) an acoustic wave propagating at the sound speed of the gas component of the fluid.

The limit $\tau_a \gg 1$ is more interesting. Here we find a weakly damped radiation-dominated acoustic wave with

$$k\approx(\omega/a)\tfrac{4}{3}[a/(\gamma-1)c\text{Bo}]^{1/2}[1-i\tfrac{3}{32}(\gamma-1)(c^2\text{Bo}/a^2\tau_a)], \tag{101.66}$$

which implies

$$v_p/a \approx \tfrac{4}{3}[a/(\gamma-1)c\text{Bo}]^{1/2} \tag{101.67}$$

and $L/\Lambda\approx[16/3\pi(\gamma-1)](a^2/c^2\text{Bo})\tau_a \gg 1$; and a strongly damped, slow, radiation diffusion wave with

$$k\approx(\omega/a)(a/c\text{Bo})[8\gamma\tau_a\text{Bo}/3(\gamma-1)]^{1/2}(1-i), \tag{101.68}$$

which implies $v_p/a\approx(c\text{Bo}/a)[3(\gamma-1)/8\gamma\tau_a\text{Bo}]^{1/2}\ll 1$ and $L/\Lambda=1/2\pi$.

To appreciate (101.67) physically, recall from (101.22) that the sound speed in a radiating fluid is $a_{\text{fluid}} = [(1+\alpha)\Gamma_1/\gamma]^{1/2} a_{\text{gas}}$, where $\alpha \equiv p_{\text{rad}}/p_{\text{gas}} = [4\gamma/3(\gamma-1)](a/c\text{Bo})$. For large r, $\alpha \gg 1$ and $\Gamma_1 \to \frac{4}{3}$; hence (101.67) simply states that the radiation-dominated acoustic mode propagates at the sound speed appropriate for a radiating fluid whose pressure and energy density are dominated by radiation. The acoustic-mode phase speed obtained from numerical solutions of (101.62) for large τ_a does, in fact, agree precisely with a_{fluid} as computed from (101.22).

As shown in Figures 101.4 and 101.5, for $r \ll 1$ the material dominates the dynamical behavior of the fluid, with only one difference from the results given by the quasi-static theory: at small τ_a the fast radiation diffusion mode, found before, is transformed into a propagating radiation

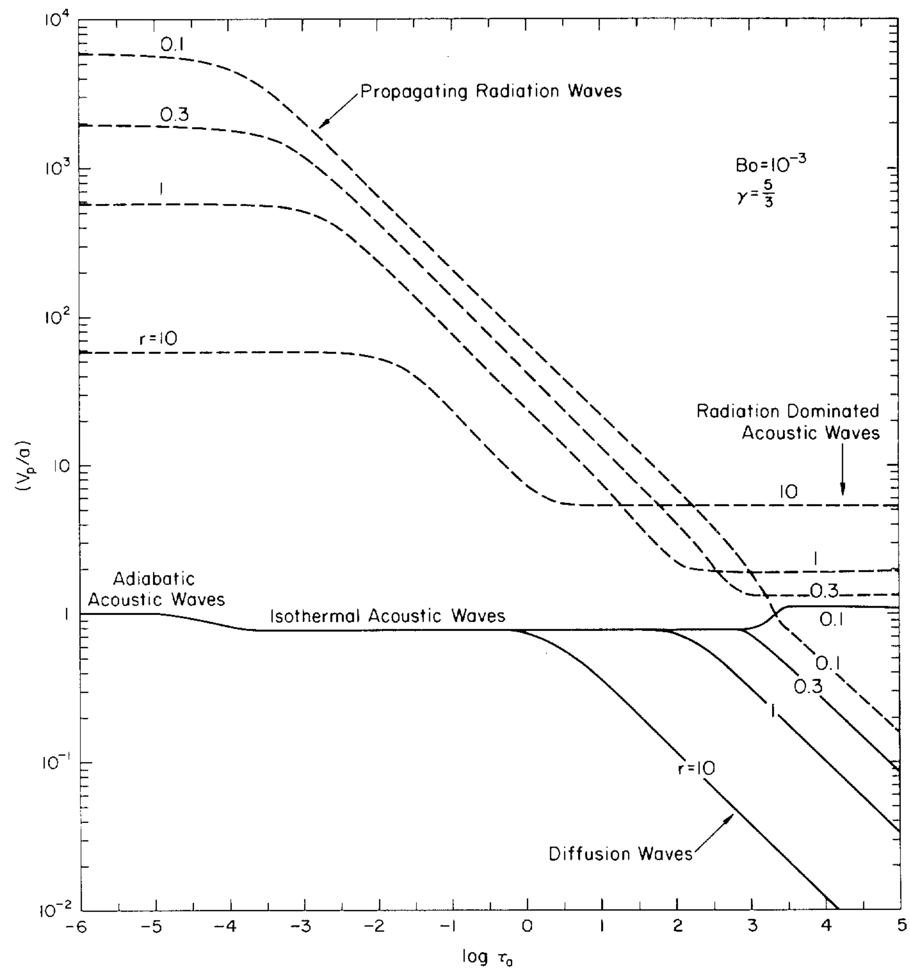

Fig. 101.4 Phase speed for acoustic, diffusion, and propagating radiation modes, allowing for time dependence and dynamical behavior of radiation field.

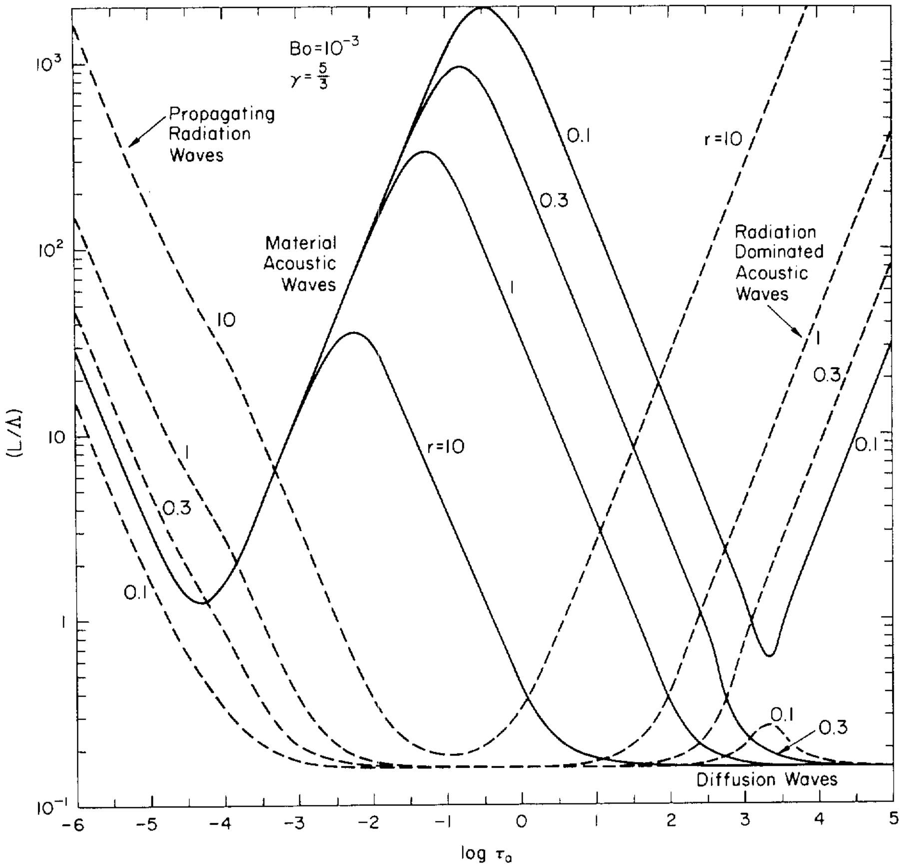

Fig. 101.5 Damping length for acoustic, diffusion, and propagating radiation modes, allowing for time dependence and dynamical behavior of radiation field.

wave as the flux-limiting properties of the time-dependent radiation equations come into play.

In contrast, for $r \geqslant 1$, the radiation field dominates the dynamics of the fluid. When τ_a is small the modes have the same behavior as for small r because the radiation and material are essentially uncoupled. But as τ_a increases it is the propagating radiation wave that merges continuously into the radiation-dominated acoustic mode, while the material acoustic mode first changes from adiabatic to isothermal and then merges continuously into the slow radiation diffusion mode.

RADIATIVE AMPLIFICATION OF ACOUSTIC WAVES

The propagation of optically thin acoustic waves in a homogeneous medium with a large radiation flux has been analyzed by Hearn (**H1**). He finds that under certain conditions the waves can be amplified by the work done by the radiation force on wave-induced variations of the opacity of the material. If the wave frequency is sufficiently low (but not so low that the disturbances become optically thick), radiative energy exchange obliterates temperature fluctuations and the wave propagates isothermally. In this case the opacity varies only in response to changes in density, hence is largest when the material is compressed, which is also when it has the greatest forward velocity. Thus the gas is most strongly accelerated when it is moving fastest in the same direction as the radiation flux, that is, when the radiation force is in phase with the velocity perturbation. Therefore the work done by radiation forces tends to increase the velocity amplitude of the wave. On the other hand, high-frequency waves are essentially adiabatic, and the decrease in opacity with increasing temperature (which occurs in hot, e.g., stellar, material) more than offsets the density-induced increase; hence these waves are damped by radiative energy losses. Unfortunately many approximations were made in this exploratory discussion, and a complete analysis using consistent Lagrangean radiation equations remains to be done.

An approximate theory describing the development of waves into the nonlinear regime under the action of this mechanism is contained in (**H2**).

102. Propagation of Acoustic-Gravity Waves in a Radiating Fluid

In this section we consider the propagation of acoustic-gravity waves in a stratified radiating atmosphere. Unfortunately relatively little work has been done on this important problem, and at present the state of the analysis is far less complete and consistent than that presented in §101 for pure acoustic waves.

WAVE DAMPING BY NEWTONIAN COOLING

Consider first the propagation of optically thin acoustic-gravity waves in which the radiative energy exchange produces Newtonian cooling. We assume the radiation field is quasi-static, hence ignore its dynamical behavior. Under these assumptions the main effect of the radiation is to damp the waves. An additional effect, as we will see below, is that we no longer obtain either pure progressive or pure standing (evanescent) waves separated crisply into distinct regions in the diagnostic diagram as in the adiabatic case discussed in §53.

(*a*) *Isothermal Atmosphere* Following Souffrin (**S16**), (**S17**) we first assume a planar isothermal atmosphere composed of a perfect gas having constant specific heats. The linearized gas energy equation (101.3) then reduces to

$$(\partial p_1/\partial t) + w_1(dp_0/dz) - a^2[(\partial \rho_1/\partial t) + w_1(d\rho_0/dz)] = -t_{RR}^{-1}[p_1 - (a^2/\gamma)\rho_1]. \tag{102.1}$$

For simplicity we assume that the radiative relaxation time t_{RR} is constant with height. Assuming that ρ_1, p_1, and w_1 are of the form (53.30) we can reduce (102.1) to

$$i\omega P[1-(i/\omega t_{RR})]-ia^2\omega R[1-(i/\gamma\omega t_{RR})]+a^2(\omega_g^2/g)W=0, \quad (102.2)$$

which differs from (53.31d) only by the imaginary terms in the coefficients of $i\omega P$ and $ia^2\omega R$. Here we used the fact that $\omega_g^2=(\gamma-1)g/\gamma H$ for a perfect isothermal gas.

Although we can again use (53.30) for P, R, W, U, and Θ, the vertical wavenumber k_z will now be complex. For this reason we replace, for the time being, ik_zW and ik_zP in (53.31a) and (53.31c) by $-(dW/dz)$ and $-(dP/dz)$, and combine those equations with (102.2) to obtain the following differential equation for W:

$$\begin{aligned}\{(\omega^2/a^2)-[1-(\omega_g^2/\omega^2)]k_x^2-(1/4H^2)+(d^2/dz^2)\\ -(i/\gamma\omega t_{RR})[(\gamma\omega^2/a^2)-k_x^2-(1/4H^2)+(d^2/dz^2)]\}W=0\end{aligned} \quad (102.3)$$

or

$$\{h_0+(d^2/dz^2)-(i/\gamma\omega t_{RR})[h_0+(d^2/dz^2)+(\gamma-1)(\omega^2/a^2)-(\omega_g^2/\omega^2)k_x^2]\}W=0. \quad (102.4)$$

Identical equations hold for P or R. In the isothermal, adiabatic limit $h_0=k_z^2$ [see (54.89)].

Again following Souffrin we note that in general we can write $k_z=k_R+ik_I$ and, assuming $W\propto\exp(-ik_zz)$,

$$(d^2W/dz^2)\equiv-(h_R+ih_I)W=-(k_R+ik_I)^2W=-[(k_R^2-k_I^2)+2ik_Ik_R]W. \quad (102.5)$$

Then substituting $-(h_R+ih_I)W$ for (d^2W/dz^2) in (102.4) we find that h_R and h_I are given by

$$h_I=\gamma\omega t_{RR}(1+\gamma^2\omega^2t_{RR}^2)^{-1}[(\omega_g^2/\omega^2)k_x^2-(\gamma-1)(\omega^2/a^2)] \quad (102.6)$$

and

$$\begin{aligned}h_R=[(\omega_g^2/\omega^2)-1]k_x^2+a^{-2}(\omega^2-\omega_a^2)\\ -(1+\gamma^2\omega^2t_{RR}^2)^{-1}[(\omega_g^2/\omega^2)k_x^2-(\gamma-1)(\omega^2/a^2)],\end{aligned} \quad (102.7)$$

where, as in §52, $\omega_a\equiv a/2H$. The real and imaginary parts of k_z are determined from $h_I=2k_Rk_I$ and $h_R=k_R^2-k_I^2$, which yield

$$k_R^2=\tfrac{1}{2}[h_R+(h_R^2+h_I^2)^{1/2}] \quad (102.8)$$

and

$$k_I^2=\tfrac{1}{2}[-h_R+(h_R^2+h_I^2)^{1/2}]. \quad (102.9)$$

The positive sign was chosen for the radical to make both k_R^2 and k_I^2 positive (i.e., k_R and k_I real) whether h_R is positive or negative.

There remains an ambiguity about which sign of $(k_R^2)^{1/2}$ and $(k_I^2)^{1/2}$ to choose. Souffrin imposes the requirement that the energy flux be positive in the positive z direction, that is, he requires that energy be carried upward from a source. The vertical component of the energy flux is given by

$$(\phi_w)_z = \tfrac{1}{4}(P^* W + PW^*) = \frac{WW^* \omega k_R(\omega^4 - g^2 k_x^2)}{\omega^4\{k_R^2 + [k_I - (1/2H) + (gk_x^2/\omega^2)]^2\}} \tag{102.10}$$

which is positive if and only if

$$\omega k_R(\omega^4 - g^2 k_x^2) > 0. \tag{102.11}$$

If we multiply both sides of $h_I = 2k_R k_I$ by $\omega k_R(\omega^4 - g^2 k_x^2)$ and use (102.6) we find

$$\begin{aligned} k_R k_I \omega(\omega^4 - g^2 k_x^2) &= \tfrac{1}{2} h_I \omega(\omega^4 - g^2 k_x^2) \\ &= -[\gamma t_{RR} \omega_g^2 / 2g^2(1 + \gamma^2 \omega^2 t_{RR}^2)](\omega^4 - g^2 k_x^2)^2 < 0, \end{aligned} \tag{102.12}$$

which is negative because both factors in the right-most expression are intrinsically positive. Comparing (102.11) and (102.12) we conclude that $k_I \leq 0$. Moreover we see that gravity waves, for which $g^2 k_x^2 > \omega^4$, have $k_R < 0$ for upward propagation of energy, whereas acoustic waves, for which $\omega^4 > g^2 k_x^2$, have positive k_R, which was also the case for adiabatic acoustic-gravity waves as discussed in §54.

From (102.8) and (102.9) we see that when $h_R = 0$, $|k_R| = |k_I|$; when $h_R > 0$, $|k_R| > |k_I|$; and when $h_R < 0$, $|k_R| < |k_I|$. Thus when $h_R < 0$, the waves are heavily damped over a single vertical wavelength, and were classified by Souffrin as *mainly damped* or *mainly evanescent*, whereas waves with $h_R > 0$ are classified as *mainly propagating*. The boundaries separating these propagating and evanescent regions in the diagnostic diagram are defined by the curves $h_R = 0$, and are shown in Figure 102.1 for several values of t_{RR}, ranging from 0 (isothermal) to ∞ (adiabatic). Damped, propagating acoustic waves lie above the upper curve, which asymptotes at small values of k_x to

$$\omega_{aN}^2(t_{RR}) = \tfrac{1}{2}[\![\omega_a^2 - (1/\gamma t_{RR}^2) + \{[\omega_a^2 - (1/\gamma t_{RR}^2)]^2 + (4\omega_a^2/\gamma^2 t_{RR}^2)\}^{1/2}]\!], \tag{102.13}$$

which is the effective acoustic cutoff frequency in the Newtonian cooling approximation. Note that ω_{aN} varies with t_{RR}. The curve bounding the region of propagating low-frequency waves asymptotes at large k_x to

$$\omega_{gN}^2(t_{RR}) = \omega_g^2 - (1/\gamma^2 t_{RR}^2), \tag{102.14}$$

the effective gravity-wave cutoff frequency in the Newtonian cooling approximation. Thus if $\omega_g < (1/\gamma t_{RR})$, gravity waves cannot propagate even when the atmosphere is convectively stable. Equation (102.14) is modified

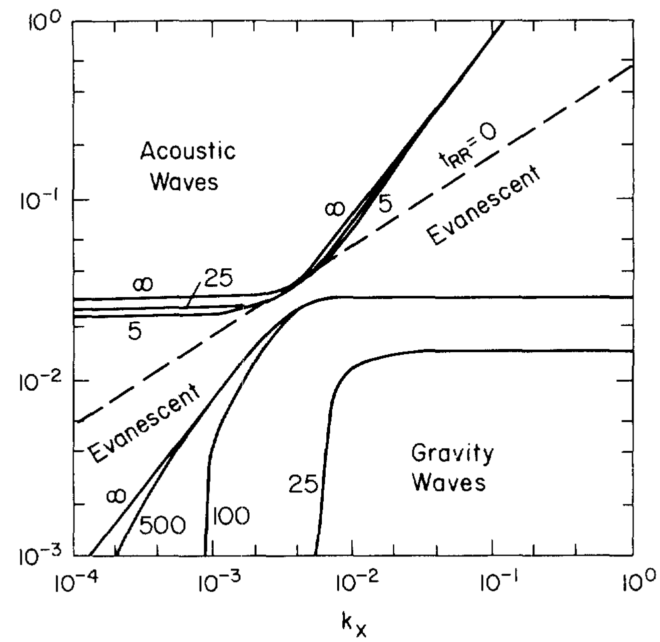

Fig. 102.1 Diagnostic diagram showing boundary curves between mostly propagating acoustic waves (upper left), mainly evanescent waves, and mostly propagating gravity waves (lower right), labeled by the value of t_{RR}.

when ω is allowed to be complex, and Stix finds (**S24**) that an impulsively generated packet of gravity waves can propagate if $\omega_g > (1/2\gamma t_{RR})$, a slightly less stringent condition.

In the expression for the energy flux, equation (102.10), WW^* is not constant with z but contains the factor $\exp(-2|k_I|z)$, hence the wave energy flux diminishes with increasing height. When $|k_I|$ is large, the decrease is very rapid. In the limit of instantaneous temperature smoothing, that is, when $t_{RR} \to 0$, we find $h_I \to 0$ and

$$h_R \to (\gamma - 1)(\omega^2/a^2) - (\omega_a^2/a^2) - k_x^2, \qquad (102.15)$$

which yields the dispersion relation for propagating isothermal sound waves; in this limit gravity waves are absent altogether.

Physically these results are not surprising, because (in the absence of gradients in the composition of the gas) the buoyancy force that drives gravity waves arises solely from horizontal temperature fluctuations, which vanish when $t_{RR} \to 0$. In contrast, acoustic waves are driven by pressure gradients whether there are associated temperature perturbations or not.

The polarization relations (53.32) are also modified by radiative energy exchange, and can change drastically when t_{RR} is small. Defining $\alpha \equiv (\gamma\omega t_{RR})^{-1}$ and $r \equiv (\gamma\omega^2 - a^2k_x^2)/(\omega^2 - a^2k_x^2)$, we find the polarization relations for waves in an isothermal atmosphere under the Newtonian cooling approximation are

$$P = \frac{\omega a^2(1+ir\alpha)}{(\omega^2 - a^2k_x^2)(1+r^2\alpha^2)}\left\{k_R + \alpha\left(k_I - \frac{1}{2H}\right) + i\left[\left(\frac{\gamma-2}{2\gamma H}\right) + k_I - \alpha k_R\right]\right\}W, \tag{102.16}$$

$$\begin{aligned} R = \frac{\omega(1+ir\alpha)}{(\omega^2 - a^2k_x^2)(1+r^2\alpha^2)}\Bigg\{&k_R + \gamma\alpha\left(k_I - \frac{1}{2H}\right) \\ &+ \frac{i}{H}\left[\left(\frac{\gamma-1}{\gamma}\right)\left(\frac{a^2k_x^2}{\omega^2}\right) - \frac{1}{2}\right] + i(k_I - \gamma\alpha k_R)\Bigg\}W, \end{aligned} \tag{102.17}$$

and, noting that $(T_1/T_0) = (p_1/p_0) - (\rho_1/\rho_0)$ implies $\Theta = (\gamma P/a^2) - R$,

$$\Theta = \frac{\omega(\gamma-1)(1+ir\alpha)}{(\omega^2 - a^2k_x^2)(1+r^2\alpha^2)}\left\{k_R + ik_I + \frac{i}{H}\left[\frac{1}{2} - \frac{1}{\gamma}\left(\frac{a^2k_x^2}{\omega^2}\right)\right]\right\}W. \tag{102.18}$$

The leading real and imaginary terms in (102.16) to (102.18) are the same as in (53.32). The quantity r is of the order unity except when ω^2 or $\gamma\omega^2$ is nearly equal to $a^2k_x^2$, while α can range from very small values (for nearly adiabatic propagation) to very large values when $2\pi\gamma t_{RR}$ is much less than a wave period.

(*b*) *Solar Model Atmosphere* In a nonisothermal atmosphere, use of the Newtonian cooling approximation provides a simple but, unfortunately, inconsistent method for studying the interaction between linear waves and the radiation field. Some of the inconsistency arises from the fact that the model atmospheres [e.g., HSRA (**G3**) or VAL (**V4**), (**V5**)] chosen to represent the ambient medium in which the waves propagate are not in radiative equilibrium. Because the physical mechanisms that determine the temperature structure of the solar atmosphere are not actually known, we have little choice but to include an unspecified nonradiative source–sink term in the gas energy equation and write

$$\frac{Dp}{Dt} - a^2\frac{D\rho}{Dt} = (\Gamma_3 - 1)\left[4\pi\int_0^\infty \kappa_\nu(J_\nu - S_\nu)\,d\nu - \nabla\cdot\mathbf{F}_{nr}\right] \tag{102.19}$$

Here $\mathbf{F}_{nr}$ represents some sort of nonradiative energy flux chosen such that $\nabla\cdot\mathbf{F}_{nr}$ exactly balances the net radiative gains and losses in the static atmosphere, that is,

$$4\pi\left[\int_0^\infty \kappa_\nu(J_\nu - S_\nu)\,d\nu\right]_0 = (\nabla\cdot\mathbf{F}_{nr})_0. \tag{102.20}$$

Because we do not know how to write $\mathbf{F}_{nr}$, we cannot do more than guess at how a wave-induced perturbation $(\mathbf{F}_{nr})_1$ would depend on T_1 and ρ_1. Therefore, in the linearized gas energy equation we have no choice but to ignore this term altogether.

A second problem is that whenever the departure from radiative equilibrium in the ambient atmosphere is large, the term $\int \kappa_{\nu 1}(J_\nu - S_\nu)_0\,d\nu$, which is ignored in the Newtonian cooling approximation, can be large and important (cf. §100). Furthermore, in the Newtonian cooling formulation, the net radiative gain term $4\pi \int \kappa_{\nu 0}(J_\nu - S_\nu)_1\, d\nu$ is approximated in terms of the *local* cooling time, which is derived as if at each height the atmosphere were optically thin over a wavelength and infinite, isotropic, homogeneous, and isothermal at the local temperature. The cooling time is then calculated from (100.17) and (100.10) using the local values of $T(z)$, $\rho(z)$, $\kappa(z)$, etc. at each height; except in the low photosphere $(\kappa_0/k) \approx 0$. However as can be seen from Figure 54.1, each of the assumptions underlying (100.17) is poor in some region of the atmosphere, with the most serious errors occurring at continuum optical depths $\tau_c \sim 10^{-2}$ to 1, where both $T(z)$ and $\kappa(z)$ vary rapidly with height, and where the gas is neither very optically thick nor optically thin. Unfortunately this is also the region where the radiative damping effects are the most important.

Retracing the arguments of §100 it is clear that (100.17) is a poor approximation to the solution of (100.7) for a highly inhomogeneous and anisotropic medium. In particular, in the solar photosphere and chromosphere the perturbations of J are essentially determined by the perturbations in S at an optical depth of about unity, not locally. Hence the nonlocally driven part of J_1 (which is lost in the local cooling-time formulation) may dominate over the locally driven part.

In treating the propagation of linear acoustic-gravity waves in the solar atmosphere we can take into account the variations of temperature, density, sound speed, buoyancy frequency, and ionization properties of the atmosphere. The resulting variations with height of the real and imaginary parts of the vertical "wavenumber" of a wave of given ω and k_x imply height-dependent variations in all properties of the wave. To describe radiative-exchange effects the best treatments available all use the Newtonian cooling approximation despite the criticisms we have just leveled at it; we merely caution the reader to remember the caveats expressed above when evaluating the results of this work. Whether the results obtained are even qualitatively correct can be determined ultimately only by computations that treat the radiation field self-consistently with the fluid equations.

If we assume the density to be fixed by the requirement of hydrostatic equilibrium, then, as in §54, the density is given by (54.75) with $H(z)$ defined by (54.68). [See (**M8**) for a discussion about $\rho(z)$, H, $\omega_{\rm BV}^2$, and a when a "turbulent pressure" is included in the model.] The amplitude functions are again as in (54.77) with $E(z)$ defined by (54.76), and the Brunt–Väisälä frequency $\omega_{\rm BV}(z)$ is given by (54.67), $H_\rho(z)$ by (54.69), and $\Gamma_1(z)$ by (14.19).

The linearized continuity and momentum equations are unchanged from their adiabatic forms (54.78a) and (54.78b). The linearized energy equation, written in terms of amplitude functions, becomes

$$i\omega[1-(i/\omega t_{RR})]P(z) - i\omega a^2[1-(i/\Gamma\omega t_{RR})]R(z) + (a^2\omega_{\rm BV}^2/g)\,W(z) = 0, \tag{102.21}$$

or

$$i\omega(1-i\alpha\Gamma)P(z)-i\omega a^2(1-i\alpha)R(z)+(a^2\omega_{BV}^2/g)W(z)=0; \quad (102.22)$$

here $\alpha\equiv(\Gamma\omega t_{RR})^{-1}$ and $\Gamma\equiv c_p/c_v$ including ionization effects as in (14.15) and (14.18). Combining (102.22) with (54.78a) and (54.78b) yields the following rather unwieldy equation for $P(z)$:

$$\begin{aligned}&\left\{\left(\frac{\omega_{BV}^2}{\omega^2}-1\right)k_x^2+\frac{\omega^2}{a^2}-\frac{1}{4H^2}+\left(\frac{d^2}{dz^2}\right)\right.\\&\left.\quad-i\alpha\left[\frac{\Gamma\omega^2}{a^2}-k_x^2-\frac{1}{4H^2}+\left(\frac{d^2}{dz^2}\right)+\left(\frac{1}{H}-\frac{1}{H_p}\right)\left(\frac{d}{dz}+\frac{1}{2H}\right)\right]\right\}P(z)\\&+\left\{\frac{d}{dz}\left(\frac{1}{2H}\right)-\frac{d\ln a^2}{dz}\left(\frac{d}{dz}-\frac{1}{2H}\right)\right.\\&\quad-i\alpha\left[\frac{d}{dz}\left(\frac{1}{2H}\right)-\left(\frac{d\ln a^2}{dz}+\frac{d\ln\alpha}{dz}\right)\left(\frac{d}{dz}-\frac{1}{2H}\right)\right.\\&\quad\left.\left.-\frac{ig\Gamma}{a^2}\left(\frac{d\ln\alpha}{dz}+\frac{d\ln\Gamma}{dz}\right)\right]\right\}P(z)=0.\end{aligned} \quad (102.23)$$

The expression in the first set of braces is identical to that in (102.3) except that ω_{BV}^2 and Γ replace ω_g^2 and γ, and a new term, which is identically zero in an isothermal medium where $H\equiv H_p$, has appeared. The second set of braces contains the derivative terms that arise because all the atmospheric properties now vary with height. The terms containing derivatives of a^2, ω_{BV}^2, Γ, and $(1/2H)$ are not exactly the same in the corresponding equations for $W(z)$ and $R(z)$.

To obtain numerical results for a realistic model atmosphere it is easier not to use (102.23) but to either (1) solve (102.22), (54.78a), and (54.78b) simultaneously at all depth points or (2) regard the atmosphere as a set of thin layers, chosen such that no atmospheric property changes much through a layer, and that the layer thickness is small compared to a vertical wavelength of the wave being studied. Within each layer all atmospheric properties are taken to be constant, but the derivatives (dT/dz) and $(d\mu/dz)$ obtained from the model are used to calculate H and ω_{BV}^2 (which in turn are taken to be constant within a layer).

We will describe results obtained by using method (2). All terms in the first set of braces in (102.23) were calculated as they would be for a continuous model; the value for each layer was then chosen to be either the midpoint of the layer or at an interface. All terms in the second set of braces were assumed to be negligible and were dropped. Because the approximations inherent in the original equations imply that at best only qualitative results will be obtained, the small inaccuracies incurred in this implementation of the layer method are not important.

We will discuss only gravity waves, for which the region of propagation lies above about 100 km, where $t_{RR}>1/\Gamma\omega_{BV}$ [cf. (102.14)]. The adopted radiative relaxation times are given by the linear relation shown in Figure 102.2, which closely approximates the relaxation times computed by Stix

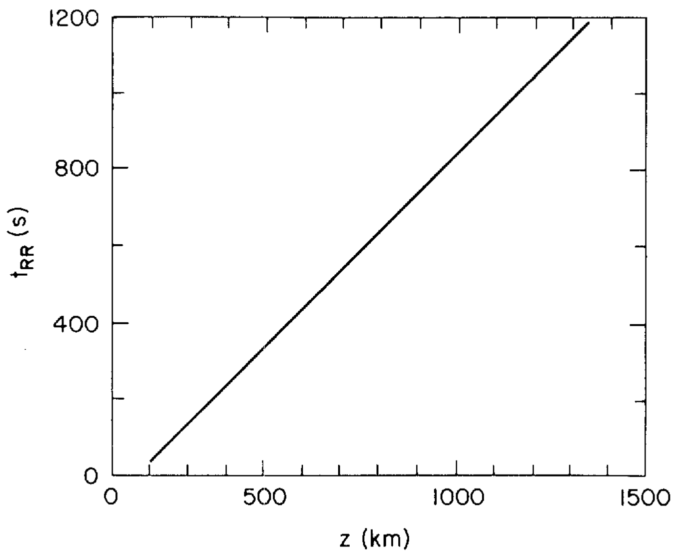

Fig. 102.2 Representative run of radiative relaxation time in a model solar atmosphere.

(**S24**) in the region 100 to 500 km, and provides a continuous extension into the chromosphere where radiative relaxation times are not known very accurately (cf. §100). For the smallest values of t_{RR} that still exceed $(\Gamma\omega_{BV})^{-1}$, the real part of k_z is much smaller than its adiabatic value, hence the vertical wavelength of the wave increases and the group velocity decreases. The change is greatest for waves that already have small values of k_z in the adiabatic limit.

Energy loss occurs most rapidly from gravity waves with large k_z, that is, from waves with large k_x for a given ω, or with small ω for a given k_x. This fact is evident in Figure 102.3, where waves A to D all have $\Lambda_x = 2000$ km and form a sequence of decreasing frequency, whereas waves D to G have a fixed period of 500 s, and form a sequence with decreasing k_x. The least damped waves are A, B, and G, while the energy flux decreases most rapidly with height for D.

Recalling (53.24b) for the vertical energy flux, we see that the energy flux can decrease both as a result of a decrease in the amplitudes of p_1 and w_1, and from an increase in the phase lag $|\delta_{PW}|$ toward $\pi/2$. The phase lags and relative amplitudes of the perturbations can be found from the polarization relations, which are now given by

$$P = \frac{ia^2\omega(\kappa_1 + i\alpha\kappa_2)}{(\kappa_1^2 + \alpha^2\kappa_2^2)}\left[\frac{\omega_{BV}^2}{g} - (1 - i\alpha)\left(\frac{1}{2H} - \frac{d}{dz}\right)\right]W, \qquad (102.24)$$

$$R = \frac{i\omega(\kappa_1 + i\alpha\kappa_2)}{(\kappa_1^2 + \alpha^2\kappa_2^2)}\left[\frac{a^2\omega_{BV}^2 k_x^2}{g\omega^2} - (1 - i\Gamma\alpha)\left(\frac{1}{2H} - \frac{d}{dz}\right)\right]W, \qquad (102.25)$$

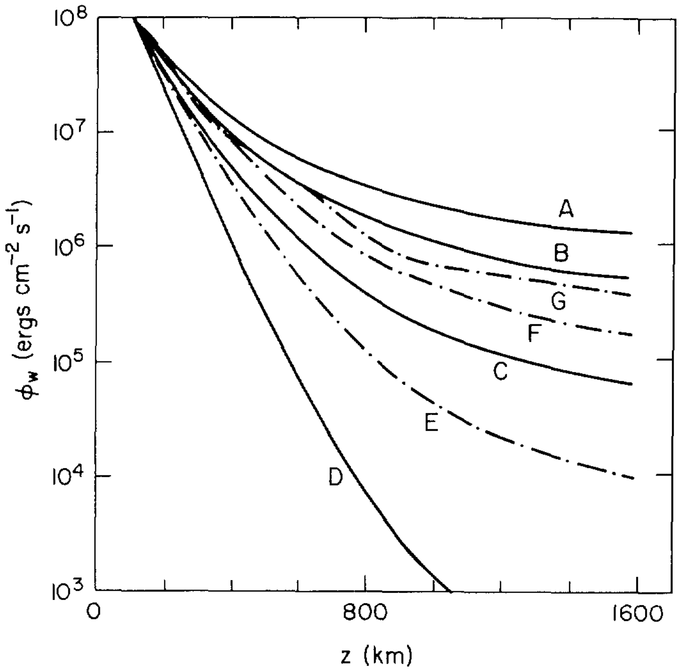

Fig. 102.3 Height dependence of energy flux for radiatively damped gravity waves with horizontal wavelength 2000 km and periods of 700, 800, 1000, and 1500 s (curves *A* to *D*), and for gravity waves with period 1500 s and horizontal wavelengths 2000, 3000, 5000, and 7000 km (curves *D* to *G*).

and

$$\Theta=\frac{1}{Q}\left(\frac{\Gamma}{a^2}P-R\right)=\frac{i\omega(\kappa_1+i\alpha\kappa_2)}{(\kappa_1^2+\alpha^2\kappa_2^2)}\left[\frac{\omega_{\rm BV}^2\kappa_2}{g\omega^2}+(\Gamma-1)\left(\frac{d}{dz}-\frac{1}{2H}\right)\right]W, \quad (102.26)$$

where $\kappa_1\equiv\omega^2-a^2k_x^2$, $\kappa_2\equiv\Gamma\omega^2-a^2k_x^2$, and Q is given by (14.33).

When $\alpha>1$, the terms $(\kappa_1+i\alpha\kappa_2)$ and $(1-i\alpha)$ are dominated by the imaginary part. In the low photosphere (102.14) implies that t_{RR} must be greater than about 25 s for gravity waves to propagate, and we find that $\alpha>1$ for $25\text{ s}<t_{RR}<100\text{ s}$ if the wave period is 600 s, and $\alpha>1$ for $25\text{ s}<t_{RR}\lesssim170\text{ s}$ if the period is 1000 s. Each of the complex terms, and $(d/dz)\rightarrow-ik_{zR}+k_{zI}$, can thus show a phase change from the adiabatic value that approaches $\pi/2$ when $t_{RR}\approx25$ s, particularly if the wave period is large.

The polarization relation for Θ is the simplest of the three, especially when the vertical wavenumber is small; then the phase lag is determined mainly by $i\omega(\kappa_1+i\alpha\kappa_2)\kappa_2$. For $\omega^2\ll a^2k_x^2$, $\kappa_1\approx\kappa_2\approx-a^2k_x^2$ and $\Theta\approx|C|\,i(1+i\alpha)W$, hence $\sin\delta_{TW}\approx(1+\alpha^2)^{-1}$, $\cos\delta_{TW}\approx-\alpha(1+\alpha^2)^{-1}$, and $\tan\delta_{TW}\approx-1/\alpha$. Thus δ_{TW} tends to $\pi/2$ as $\alpha\rightarrow0$ (adiabatic) and $\delta_{TW}\rightarrow\pi$

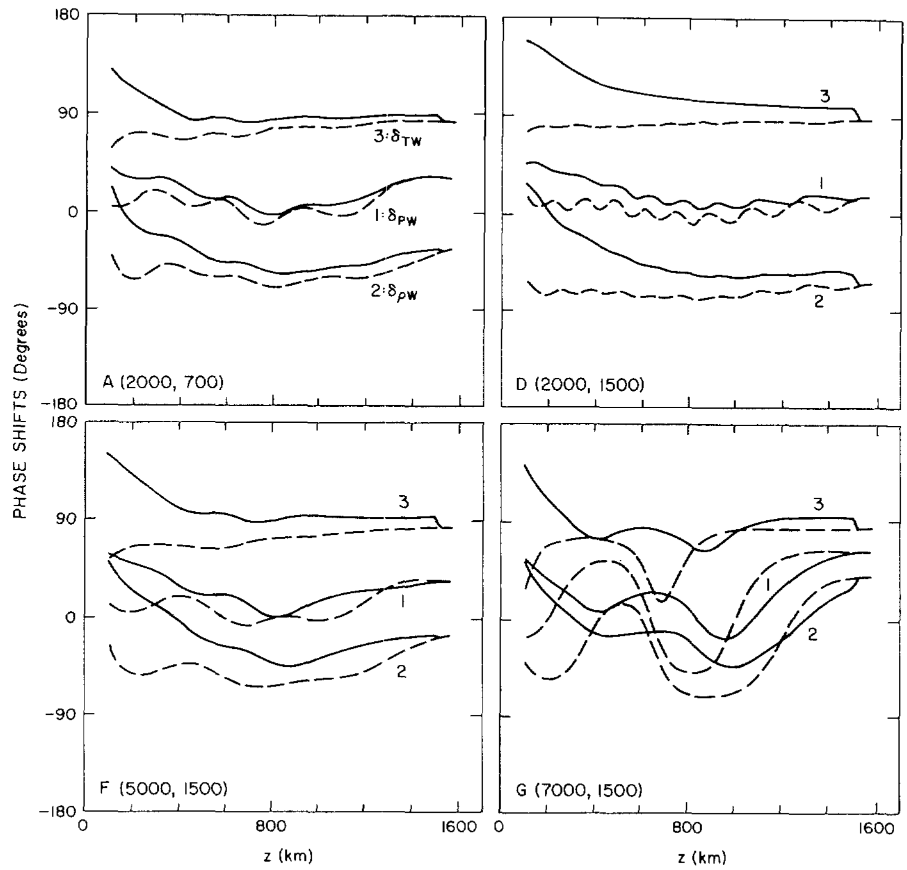

Fig. 102.4 Phase shifts for gravity waves in model solar atmosphere. (1) δ_{PW}. (2) $\delta_{\rho W}$. (3) δ_{TW}. *Solid curves* allow for radiative damping in Newtonian cooling approximation. *Dashed curves* are for adiabatic waves.

as $\alpha \to \infty$ (strong radiative damping). Precisely this behavior is seen in Figure 102.4, where the adiabatic and radiatively damped phase lags are compared for several waves.

The polarization relations for P and R are more complicated, each containing three terms that are real in the adiabatic limit and complex when radiative damping occurs. The values of δ_{RW} and δ_{PW} in Figure 102.4 for radiatively damped waves both reflect the interplay of the three complex terms, as well as some interference (much less than in the adiabatic case) between upward-propagating and reflected waves.

WAVE DAMPING BY RADIATIVE TRANSPORT

Near the open boundary of a radiating medium such as a stellar atmosphere the radiation terms in the energy equation are strongly nonlocal, and should be determined from the transfer equation. To study the radiative damping of acoustic-gravity waves in a realistic model solar atmosphere, Schmieder (**S5**), (**S6**) coupled the linearized fluid equations to a linearized transfer equation. Because the perturbation $J_{\nu 1}$ of the radiation field at each depth then depends on T_1 at all depths, the fluid equations at one depth become globally coupled to those at all other depths.

Several approximations were made to simplify the equations. First, the waves were assumed to propagate in the vertical direction only, hence $k_x \equiv 0$ and the problem becomes strictly one dimensional. Second, LTE was assumed, so all material properties can be written as functions of ρ and T. Third, the perfect gas law was used and ionization effects are neglected. Fourth, the radiation field was assumed to be quasi-static, hence its dynamical behavior was ignored. Schmieder analyzed waves in the region from an optical depth of about 3 up to a height of 500 km, that is, the top of the photosphere. The assumption of linearity gradually breaks down with increasing height, but should be quite good in the lower layers where radiative damping is most important.

The run of temperature and density was taken from the HSRA model solar atmosphere (**G3**). The ambient atmosphere is thus neither in radiation nor in hydrostatic equilibrium. The linearized continuity and momentum equations are given by (52.24) and (52.25) (ignoring the non-gravitational force term that is implicit in the HSRA model). These can be rewritten in terms of the vertical displacement ζ_1, defined by $w_1 = (\partial \zeta_1/\partial t)$. Thus integrating (52.24) over time we have

$$\rho_1 + \zeta_1(d\rho_0/dz) + \rho_0(\partial \zeta_1/\partial z) = 0, \tag{102.27}$$

while (52.25) becomes

$$\rho_0(\partial^2 \zeta_1/\partial t^2) = -(\partial p_1/\partial z) - \rho_1 g. \tag{102.28}$$

The gas-energy equation (ignoring the nonradiative term that drives the ambient atmosphere out of radiative equilibrium) becomes

$$\begin{aligned} \frac{p_0}{\rho_0 T_0}\left(\frac{\partial T_1}{\partial t} + \frac{\partial \zeta_1}{\partial t}\frac{dT_0}{dz}\right) - \frac{(\gamma-1)p_0}{\rho_0}\frac{\partial}{\partial z}\left(\frac{\partial \zeta_1}{\partial t}\right) \\ = 4\pi(\gamma-1)\left[\int_0^\infty \kappa_{\nu 0}(B_{\nu 1} - J_{\nu 1})\,d\nu + \int_0^\infty \kappa_{\nu 1}(B_{\nu 0} - J_{\nu 0})\,d\nu\right]. \end{aligned} \tag{102.29}$$

The first term on the right-hand side results from the wave-induced perturbations of the mean intensity and the local Planck function. The second term arises from two effects: (1) the perturbation of the opacity, and (2) the departure of the ambient atmosphere from radiative equilibrium, which implies that $J_0 \neq B_0$. When $J_0 - B_0$ is large, the second term, which is strongly model dependent, may also be large; thus errors in the assumed $T(z)$ of the unperturbed model may produce important errors in the radiative damping calculations.

The mean intensity is approximated by the average of the incoming and outgoing intensity calculated from the formal solution (79.14) and (79.15) for a representative ray, using the Planck function from the model atmosphere. The radiative energy exchange term in (102.29) is approximated by a quadrature sum over a few representative frequencies. Perturbed intensities are computed from the linearized formal solution, which is constructed recursively between successive depth points in a discrete mesh. To pose boundary conditions for the transfer equation, it is assumed that there is no incoming radiation at the top of the atmosphere, whereas the bottom of the atmosphere is taken deep enough that the diffusion approximation can be applied. The hydrodynamic boundary conditions are imposed at the upper boundary where the value of ζ_1 is fixed; all perturbations are normalized to this value. Initial relations between T_1 and ζ_1 at the uppermost two grid points are obtained by assuming adiabatic motion and a pure outgoing wave at the top of the atmosphere. The resulting matrix equation is solved numerically and iterated to consistency.

In Schmieder's solution the general form of the radiative exchange term at depth point i is

$$\dot{q}_i = \sum_j A_{ij} T_{1,j} + \sum_j B_{ij} \rho_{1,j}. \tag{102.30}$$

The matrix B_{ij} is almost diagonal. The matrix A_{ij}, however, clearly reveals the strong nonlocal effect of the radiation-field perturbations produced by temperature perturbations up to about one photon mean free path away from the chosen depth i. For heights above about 200 km the dominant contributions to $\dot{q}_i$ from A_{ij} come from (1) a region between about −40 km and +140 km, and (2) the local temperature perturbation, which produces the diagonal elements. The departure of the ambient atmosphere from radiative equilibrium also makes an important contribution to the diagonal elements. Below 200 km, the opacity increases exponentially, and the largest off-diagonal elements in A_{ij} become confined to a small range of depths immediately above and below the point z_i. At large optical depths, a photon mean free path becomes small compared to the grid spacing and A_{ij} becomes essentially diagonal.

Schmieder finds that the wave amplitudes increase less rapidly with height for radiatively damped waves than for adiabatic waves. Evanescent waves are less affected by radiative damping than propagating waves; therefore although the amplitude of an adiabatic evanescent wave grows more slowly with height than that of a propagating wave, the difference in growth rates between radiatively damped evanescent and propagating waves is not large. For example, for a 140-s propagating nonadiabatic wave, the relative temperature perturbation ($|T_1|/T_0\,|w_1|$) is distinctly suppressed compared to that of an adiabatic wave in the region of strong damping. For a 300 s evanescent wave, however, the relative temperature perturbation is actually larger in the nonadiabatic case than in the adiabatic case.

In the isothermal Newtonian-cooling approximation, one can show that $|T_1|/T_0\,|w_1|$ is always decreased by radiative damping, whether the wave propagates or is evanescent. Schmieder's result for the 300 s wave thus arises from a nonlocal effect. The wavelength of an evanescent wave is very long, hence the phase changes little from the bottom to the top of the photosphere. Therefore transfer of radiation from around optical depth unity to greater heights tends to enhance $|T_1|/T_0$ relative to $|w_1|$.

In a second paper (**S6**) Schmieder discusses the possibility of determining an "equivalent damping time", which could be used in the Newtonian cooling formalism, at each height in the atmosphere. She first derives at each height the damping time that would cause the displacement amplitude to decrease locally at the same rate as given by the nonlocal computation; these are shown as curves "A" in Figure 102.5 for waves of different periods. However, as we saw above for Newtonian cooling in an isothermal atmosphere, radiative damping also alters the phase relations among the perturbation variables. Schmieder derives another equivalent damping time for each wave from the changes in phase lags; these are shown as curves "B" in Figure 102.5, with error bars resulting from the uncertainties in

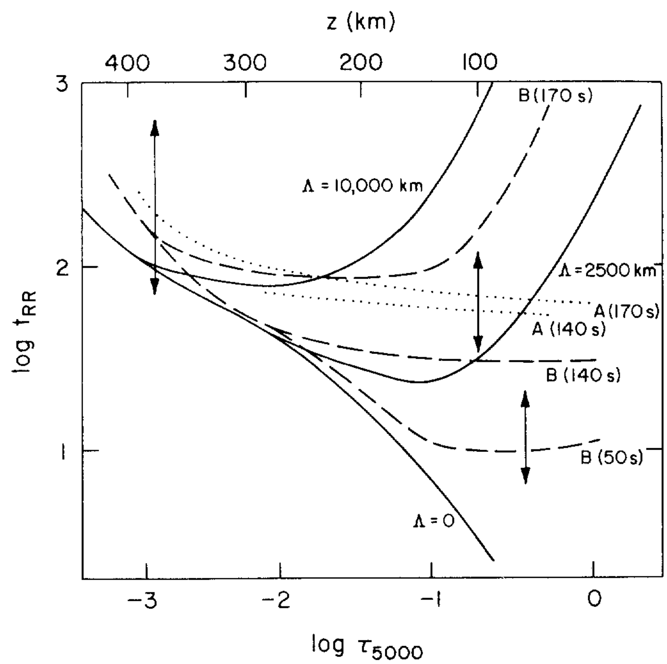

Fig. 102.5 Equivalent damping times derived from amplitude variations (dotted curves) and phase variations (dashed curves) of acoustic waves of indicated periods. Radiative damping is calculated from a solution of the transfer equations; error bars (vertical arrows) indicate uncertainties in phase curves. solid curves show local damping times computed from Spiegel's formula. From (**S6**), by permission.

assigning phases in waves that vary strongly with height. The two sets of curves give markedly different results. Had damping times been estimated from amplitude ratios, yet different results would have been obtained.

For comparison, damping times computed for waves of the same periods using Spiegel's formula (100.17) are shown in Figure 102.5 as solid curves. The important conclusion to be drawn is that nonlocal effects resulting from radiative transfer produce *qualitatively* different propagation characteristics in a wave, and that it simply is not possible to reproduce these effects self-consistently with a single equivalent damping time.

STABILITY OF ACOUSTIC-GRAVITY WAVES IN A RADIATING FLUID

The possibility of radiation-driven instabilities in a stratified radiating atmosphere has been discussed by Berthomieu et al. (**B4**) and by Spiegel (**S19**). In the former paper, it is shown that above a certain critical frequency, isothermal, optically thin perturbations in an isothermal slab of an atmosphere traversed by a radiation field can be amplified by radiation in a drift instability. In the latter paper it is shown that under certain circumstances radiation forces can drive instabilities in a stratified radiating fluid. A detailed analysis is presented for quasi-adiabatic *photoacoustic* and *photogravity* modes. The results are intriguing, but it would take us too far afield to discuss them here; the interested reader should consult the original paper.

8.2 Nonlinear Flows

103. Thermal Waves

Thermal waves result from conductive energy-transport processes within a fluid, which give rise to an energy flux $\mathbf{q} = -K\,\nabla T$. For nonradiating neutral gases it is usually satisfactory to assume *linear conduction* (K independent of T) because the conductivity depends only weakly on temperature (cf. §33). But in ionized plasmas where $K \propto T^{5/2}$, and in opaque radiating fluids where the radiation conduction coefficient depends strongly on T, we must treat *nonlinear conduction.* The distinction is important because thermal waves behave qualitatively differently in the two cases.

A problem of some interest in radiation hydrodynamics is the penetration of radiation from a hot source into cold material, a process that is reasonably well described by treating the radiation field in the diffusion approximation. Practical examples are the penetration of stellar radiation into the interstellar medium at the instant of star formation or of a supernova explosion, or the irradiation of a fusion pellet by intense laser beams. Such propagating radiation fronts are called *Marshak waves* (**M4**) or *radiation diffusion waves.*

Because radiative energy exchange is very efficient, significant radiation penetration and energy deposition can occur in a time much too short for